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Some problems of unlikely intersections in families of abelian varieties

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Introduction

The study of Diophantine equations, that is, the study of integer or rational solutions of polynomial equations with integer coefficients, is one of the most classical problems in number theory. From the perspective of algebraic geometry, such equations define algebraic varieties, and their integral or rational solutions correspond to integral or rational points on these varieties. This geometric viewpoint forms the basis of *Diophantine geometry*, which seeks to study rational solutions of Diophantine equations using the geometric structure of the associated varieties.

In dimension one, i.e. for algebraic curves, the behaviour of rational points is controlled by the genus. More precisely, if C is a smooth curve of genus g defined over a number field K, then:

- If $g = 0$, then either the set of K-rational points $C(K)$ is empty, or C is isomorphic over K to $\mathbb{P}^1$; in the latter case $C(K)$ is infinite and Zariski dense in $C(\mathbb{C})$. In particular, this is always the case after passing to a suitable finite extension of K.

- If $g = 1$, then either $C(K)$ is empty or C is an elliptic curve over K. In the second case, the Mordell–Weil theorem implies that $C(K)$ is a finitely generated abelian group. For a suitable finite extension L/K, $C(L)$ is an infinite set and, thus, Zariski dense in $C(\mathbb{C})$.

On the other hand, when $g \geq 2$, it was originally conjectured by Mordell [Mor22] that $C(K)$ is finite. Mordell's conjecture was proved by Faltings in 1983 [Fal83], following a strategy that involved reducing the problem to the Shafarevich conjecture via earlier work of Parshin.

Theorem (Faltings). *Let C be a smooth projective curve of genus $g \geq 2$ defined over a number field K. Then $C(K)$ is a finite set.*

An alternative proof of Faltings's theorem was given by Vojta [Voj91], using techniques from Diophantine approximation. Subsequent simplifications of Vojta's argument were proposed by Faltings himself [Fal91] and by Bombieri [Bom90]. More recently, a proof relying on p-adic Hodge theory was obtained by Lawrence and Venkatesh [LV20].

Another possible approach to proving Faltings's theorem begins by observing that if $C(K) \neq \emptyset$, then one can consider the curve C as embedded into its Jacobian variety J_C (via the Abel-Jacobi map), yielding an identification

$$C(K) = C \cap J_C(K) \subseteq J_C.$$

By the Mordell–Weil theorem, the group of K-rational points $J_C(K)$ is finitely generated. Thus, the set of rational points on C can be viewed as the intersection of a subvariety of an abelian variety with a finitely generated subgroup of the ambient variety, which we expect to be finite.

This perspective naturally invites generalization. One may consider more general ambient varieties, such as algebraic tori, abelian or semiabelian varieties; replace $J_C(K)$ with more general subgroups, such as the torsion points, finitely generated subgroups, or even subgroups of finite rank; and replace C with higher-dimensional subvarieties.

For example, replacing $J_C(K)$ with the set of torsion points yields the original formulation of the *Manin-Mumford conjecture*, which we will revisit in more detail in Chapter 1.

This reinterpretation of Diophantine finiteness questions in terms of intersections between subvarieties and arithmetic subsets of algebraic groups leads to a broader framework: the theory of *unlikely intersections*. The central idea is that one expects the intersection between a fixed subvariety (satisfying certain genericity assumptions) and a countable family of subvarieties with suitable properties to be non-dense, or even finite. This setting unifies several important statements in Diophantine geometry, such as the above-mentioned Manin–Mumford conjecture, the Mordell–Lang conjecture, and the André–Oort conjecture.

More specifically, given two subvarieties V, W of an ambient variety X such that $\dim X > \dim V + \dim W$, we usually expect the intersection $V \cap W$ to be empty. For this reason, if $V \cap W \neq \emptyset$, we say that the intersection is "unlikely". As mentioned before, we are interested in the case in which V is fixed and W varies in a countable family of subvarieties, known as "special subvarieties". In this setting, the *Zilber–Pink conjecture*, independently formulated in various contexts by Zilber, Pink, and Bombieri, Masser and Zannier, predicts that if V is not contained in a proper special subvariety, then its intersection with the union of special subvarieties of codimension $\geq \dim V + 1$ is not Zariski dense.

In this thesis, we investigate several instances of the Zilber–Pink conjecture, with particular focus on the setting of families of abelian varieties. After introducing the general conjecture and surveying known results in Chapter 1, we focus on two cases: curves in products of powers of elliptic schemes (Chapter 3) and curves in abelian schemes (Chapter 4). The main contributions of the thesis are Theorems 3.2 and 4.1, which establish new cases of the conjecture in these settings.

Structure of the thesis

We conclude this introduction with a brief overview of the contents of each chapter.

In Chapter 1, we provide an overview of the main results about unlikely intersections, with particular emphasis on the Zilber–Pink conjecture. We review the current state of the art, focusing on the cases of algebraic tori, abelian varieties and, more importantly, families of abelian varieties.

Chapter 2 is dedicated to height functions. We recall the definitions of Weil heights on projective varieties and canonical heights on abelian varieties, along with their main properties. We also prove the first original result of this thesis, Theorem 2.33, which provides explicit bounds for the canonical height of $f(P)$ in terms of the canonical height of P, where f is an endomorphism of an abelian variety A defined over $\overline{\mathbb{Q}}$ and $P \in A(\overline{\mathbb{Q}})$.

In Chapter 3, we study the Zilber–Pink conjecture in the case of a curve contained in a product of two fibered powers of the Legendre family. The main result is Theorem 3.2, which proves finiteness of the intersection of the curve with proper algebraic subgroups of fibers for which there are non trivial homomorphisms between the two powers. This chapter is based on the preprint [Fer24], currently under review, and is presented here with only minor changes.

In Chapter 4, we further explore the Zilber–Pink conjecture for curves in abelian schemes. The main result of this chapter, Theorem 4.1, considers the intersections of a curve with the union of all proper algebraic subgroups of the fibers with complex multiplication, extending a previous result by Barroero. The material of this chapter, together with Section 2.6, will form the basis of an article to be posted on arXiv, before submitting it to a journal.

Contents

Chapter 1

Unlikely intersections and the Zilber–Pink conjecture

This chapter introduces the topic of *Unlikely Intersections*, a class of problems in Diophantine geometry that arise from heuristic expectations on the dimension of intersections between subvarieties. The study of unlikely intersections plays a central role in Diophantine geometry, and some fundamental references on this topic are the books [Pil22] and [Zan12], which provide a comprehensive treatment of the main results, conjectures, and techniques, and the survey article [Cap23].

The central idea which motivates most of the results of this field originates from a classical result in algebraic geometry:

Theorem 1.1 (Lemma 43.13.4 (0AZP) from [Sta24]). *Let X be a smooth variety and let $V, W \subseteq X$ be closed irreducible subvarieties. Then, every irreducible component of $V \cap W$ has dimension at least* $\dim(V) + \dim(W) - \dim(X)$.

For general subvarieties $V, W \subseteq X$, one typically expects[1]

$$\dim(V \cap W) = \dim(V) + \dim(W) - \dim(X)$$

and if $\dim(V) + \dim(W) < \dim(X)$, the intersection $V \cap W$ should be empty. If, instead, $V \cap W \neq \emptyset$ despite this expectation, we say that the intersection is *unlikely*.

More generally, problems concerning unlikely intersections can be formulated as follows. Let X be an ambient variety, and let $\mathcal{F}$ be a countable collection of subvarieties of X satisfying certain properties (see [BD24], [Ull17] and [KUY15] for some examples), which we refer to as *special subvarieties*. We study the intersections of a fixed irreducible subvariety $V \subseteq X$ with the special subvarieties $W \in \mathcal{F}$ satisfying $\dim V < \operatorname{codim} W$. Heuristically, we expect $V \cap W = \emptyset$ for "most" $W \in \mathcal{F}$, unless a specific geometric or

[1] This expectation can be formalized by using suitable moduli spaces.

arithmetic relation exists between V and $\mathcal{F}$. If no such relation exists, we expect the union

$$\bigcup_{\substack{W\in\mathcal{F}\\ \dim V<\operatorname{codim} W}} (V\cap W)$$

not to be Zariski dense in V.

In this thesis we will always work over $\mathbb{C}$ (or more generally, over a field of characteristic zero), as some of the results stated in this chapter are false in positive characteristic. In the rest of this chapter we will always assume the varieties to be defined over $\mathbb{C}$, unless otherwise stated.

Example 1.2. A natural example of this setting arises in semiabelian varieties, for which we can choose $\mathcal{F}$ to be the collection of the irreducible components of the algebraic subgroups or, equivalently, the translates of semiabelian subvarieties by torsion points, also known as torsion cosets. We will explore problems in this setting in Sections 1.2 and 1.3.

One of the simplest problems in this area occurs when considering special subvarieties of dimension 0, known as *special points*. In this case, the set of special points in any special subvariety is Zariski dense, whereas for a non-special subvariety, we expect the set of special points not to be Zariski dense. In the setting of semiabelian varieties, where the special points are precisely the torsion points, this expectation is formalized by the following theorem.

Theorem 1.3 (Manin-Mumford conjecture)**.** *Let X be a semiabelian variety and $V \subseteq X$ be an irreducible subvariety. Then V contains only finitely many maximal special subvarieties. Equivalently, the set of special points in V is Zariski dense in V if and only if V is special.*

Independently proposed by Manin and Mumford in the 1960s for curves embedded in their Jacobians, the conjecture was later proven by Laurent [Lau84] for tori, by Raynaud [Ray83a, Ray83b] for abelian varieties, and by Hindry [Hin88] for semiabelian varieties. Since then, several alternative proofs have been proposed. Notably, Ullmo [Ull98] and Zhang [Zha98a] proved a stronger version conjectured by Bogomolov, extending the result to points of sufficiently small Néron–Tate height. Pila and Zannier [PZ08] provided another approach using techniques from o-minimality.

In general, these questions are typically studied in the broader context of (mixed) Shimura varieties, originally introduced by Deligne [Del71, Del79], building on special cases introduced by Shimura [Shi63] as moduli spaces of abelian varieties with additional structures. A Shimura variety naturally contains a distinguished collection of subvarieties, known as *subvarieties of Hodge type*, which serve as its special subvarieties.

In this thesis, we focus on specific examples of mixed Shimura varieties, particularly families of abelian varieties, and therefore we will not introduce the general theory. For further background, we refer to [Mil05] and [Pin05a]. Nevertheless, for the sake of generality, we will state all key conjectures in the setting of mixed Shimura varieties.

A natural generalization of the Manin-Mumford conjecture to Shimura varieties was proposed by André and Oort.

Theorem 1.4 (André-Oort conjecture)**.** *Let X be a (mixed) Shimura variety and $V \subseteq X$ be a subvariety. Then V contains only finitely many maximal special subvarieties. Equivalently, the set of special points in V is Zariski dense in V if and only if V is special.*

The André-Oort conjecture was posed independently by André [And89, Section X.4] for curves in general Shimura varieties and by Oort [Oor97] for general subvarieties of the moduli space of principally polarized abelian varieties $\mathbb{A}_g$. It was first proved by Klinger, Ullmo and Yafaev [KY14, UY14] under the Generalized Riemann Hypothesis, using ideas by Edixhoven [Edi05]. It was later proved unconditionally for $\mathbb{C}^n$ by Pila [Pil11] and for $\mathbb{A}_g$ by Tsimerman [Tsi18]. It was finally proved in full generality in 2022 by Pila, Shankar and Tsimerman [PST$^+$22], building on work of Binyamini, Schmidt and Yafaev [BSY23] and Gao [Gao16].

1.1 The Zilber–Pink conjecture

We now turn to the main conjecture in the field of unlikely intersections, the *Zilber–Pink conjecture*. It was proposed independently by Bombieri, Masser and Zannier [BMZ99] in the case of tori, by Zilber [Zil02] for semiabelian varieties and by Pink [Pin05b] in the more general setting of mixed Shimura varieties. Its statement follows the general heuristic outlined above, providing a broad formulation that encompasses most problems concerning unlikely intersection.

Conjecture 1.5 (Zilber–Pink, Version 1)**.** *Let X be a mixed Shimura variety or a semiabelian variety. For every integer $n \geq 0$, denote by $X^{[n]}$ the union of all special subvarieties of X of codimension at least n. Then, if $V \subseteq X$ is an irreducible subvariety not contained in any proper special subvariety of X,*

$$V \cap X^{[\dim V+1]}$$

is not Zariski dense in V.

This formulation, stated by Pink in [Pin05b, Conjectures 1.3 and 5.1], is sometimes referred to as *Pink's conjecture*.

Remark 1.6. In this thesis, we will deal mainly with the case in which $V \subseteq X$ is a curve. In particular, notice that in this case Conjecture 1.5 reduces to the statement that, if V is not contained in any proper special subvariety, then $V \cap X^{[2]}$ is a finite set.

In what follows, we explore equivalent formulations.

Definition 1.7. Let X be a mixed Shimura variety or a semiabelian variety and $V \subseteq X$ a subvariety. A subvariety $W \subseteq V$ is called *atypical* (for V in X) if there exists a special subvariety S such that W is an irreducible component of $V \cap S$ and

$$\dim W > \dim V + \dim S - \dim X.$$

Conjecture 1.8 (Zilber–Pink, Version 2)**.** *Let X be a mixed Shimura variety or a semiabelian variety and $V \subseteq X$ be an irreducible subvariety. Then V contains only finitely many maximal atypical subvarieties.*

Note that if V is contained in a proper special subvariety of X, then V is an atypical subvariety of itself, and so the conjecture holds trivially.

As the irreducible components of the intersection of two special subvarieties are again special, we introduce the following definition.

Definition 1.9. Let X be a mixed Shimura variety or a semiabelian variety and let $V \subseteq X$ be an irreducible subvariety. We denote by $\langle V \rangle$ the smallest special subvariety of X containing V (i.e. the intersection of all the special subvarieties containing V), which is often called the *special closure* of V.

The following definition provides a way to measure how far a subvariety is from being special.

Definition 1.10. Let X be a mixed Shimura variety or a semiabelian variety and let $V \subseteq X$ be an irreducible subvariety. We define the *defect* of V as

$$\delta(V) = \dim \langle V \rangle - \dim V.$$

We say that a subvariety $W \subseteq V$ is *optimal* for V (in X) if for every subvariety $U \subseteq X$ such that $W \subset U \subseteq V$, we have $\delta(W) < \delta(U)$.

Notice that V is clearly an optimal subvariety of itself. Moreover, if $W \subset V$ is optimal, then we must have $\delta(W) < \delta(V)$, that is

$$\dim \langle W \rangle - \dim W = \delta(W) < \delta(V) = \dim \langle V \rangle - \dim V$$

which is equivalent to

$$\dim W > \dim V + \dim \langle W \rangle - \dim \langle V \rangle$$

and this implies that W is atypical for V in $\langle V \rangle$. Conversely, W is atypical for V in X if $\delta(W) < \operatorname{codim}(V)$.

Conjecture 1.11 (Zilber–Pink, Version 3)**.** *Let X be a mixed Shimura variety or a semiabelian variety and $V \subseteq X$ be an irreducible subvariety. Then V contains only finitely many optimal subvarieties.*

Although these formulations appear different, Conjectures 1.5, 1.8 and 1.11 are in fact equivalent, provided that each of them is assumed to hold for all ambient varieties (and not just for a fixed instance). The equivalence of Conjectures 1.5 and 1.8 is proved in Section 12 of [BD24], while Lemma 2.7 of [HP16] proves the equivalence of Conjectures 1.8 and 1.11.

Remark 1.12. A subvariety $S \subseteq X$ is special if and only if $\delta(S) = \dim \langle S \rangle - \dim S = 0$. Thus, any maximal special subvariety contained in V is optimal. Therefore, the Zilber–Pink conjecture implies both the Manin-Mumford conjecture and the André-Oort conjecture (Theorems 1.3 and 1.4).

1.2 Unlikely intersections in multiplicative groups

Having outlined the general formulation of the Zilber–Pink conjecture, we now examine specific instances in particular settings, starting with algebraic tori. The case of the multiplicative group $\mathbb{G}_m^n$ has been extensively studied, leading to a series of important partial results that provide evidence for the conjecture in this context.

A prototypical example, and one of the first problems in this area, was posed by Lang in the 1960s and independently proved by Ihara, Serre, and Tate (see [Lan65] for an account of these proofs).

Theorem 1.13. *Let $f(X, Y) \in \mathbb{C}[X, Y]$ be an irreducible polynomial such that there exist infinitely many pairs of roots of unity (μ_1, μ_2) satisfying $f(\mu_1, \mu_2) = 0$. Then, up to scaling, f has the form*

$$X^n Y^m - \zeta \qquad \text{or} \qquad X^n - \zeta Y^m$$

where $n, m \in \mathbb{N}$ are not both zero and ζ is a root of unity.

As discussed in Example 1.2, the points (μ_1, μ_2), where μ_1, μ_2 are roots of unity, correspond precisely to the special points of $\mathbb{G}_m^2$. Similarly, the curves defined by the equations $X^n Y^m = \zeta$ and $X^n = \zeta Y^m$, where ζ is a root of unity, are special subvarieties of $\mathbb{G}_m^2$ since they are translates of algebraic subgroups by torsion points.

Thus, we can naturally interpret this theorem as follows: an irreducible curve $C \subseteq \mathbb{G}_m^2$, defined by an equation $f(X, Y) = 0$, contains infinitely many special points if and only if it is itself a special subvariety. In other words, this result is a direct instance of the Manin-Mumford conjecture (Theorem 1.3) for $\mathbb{G}_m^2$, which, for $\mathbb{G}_m^2$, is equivalent to the Zilber–Pink conjecture.

Regarding the Zilber–Pink conjecture for $\mathbb{G}_m^n$, we recall that $(\mathbb{G}_m^n)^{[d]}$ denotes the union of all special subvarieties, i.e., the torsion cosets, of codimension at least d. In [BMZ99], Bombieri, Masser, and Zannier proved that if $V \subseteq \mathbb{G}_m^n$ is an irreducible curve defined over $\overline{\mathbb{Q}}$ and not contained in a translate of a proper algebraic subgroup, then

$$V \cap (\mathbb{G}_m^n)^{[2]} = \bigcup_{\operatorname{codim} H \geq 2} (V \cap H),$$

where the union is taken over all torsion cosets of codimension at least 2, is finite (see also [CMPZ16] for an alternative proof using o-minimality). Note that Conjecture 1.5 suggests that this finiteness should still hold under the weaker assumption that V is not contained in a *torsion* translate of a proper algebraic subgroup.

Conjecture 1.5 for curves in $\mathbb{G}_m^n$ defined over $\overline{\mathbb{Q}}$ was later established by Maurin [Mau08].

Theorem 1.14 (Maurin)**.** *Let $V \subseteq \mathbb{G}_m^n$ be an irreducible curve, defined over $\overline{\mathbb{Q}}$ and not contained in any proper torsion coset of $\mathbb{G}_m^n$. Then, $V \cap (\mathbb{G}_m^n)^{[2]}$ is a finite set.*

In the same year, Bombieri, Masser, and Zannier [BMZ08b] extended Maurin's result to curves defined over $\mathbb{C}$. Additionally, in collaboration with Habegger, they provided an alternative proof of Theorem 1.14 in [BHMZ10].

Notice that, in the case of hypersurfaces in $\mathbb{G}_m^n$, the Zilber–Pink conjecture is equivalent to Theorem 1.3. Beyond the case of curves, significant progress has also been made in studying subvarieties of codimension 2, with Bombieri, Masser, and Zannier [BMZ07] proving the following result.

Theorem 1.15 (Bombieri-Masser-Zannier)**.** *Let $V \subseteq \mathbb{G}_m^n$ be an irreducible subvariety of dimension $n-2$, defined over $\overline{\mathbb{Q}}$ and not contained in any torsion coset of $\mathbb{G}_m^n$. Then, $V \cap (\mathbb{G}_m^n)^{[n-1]}$ is not Zariski dense in V.*

However, with the exception of some results for planes [BMZ08a], the general case of surfaces in $\mathbb{G}_m^n$ remains open. Nonetheless, there have been several important partial results for general subvarieties of $\mathbb{G}_m^n$.

Definition 1.16. Let G be a semiabelian variety and $V \subseteq G$. An irreducible subvariety $W \subseteq V$ is called *anomalous* if there exists a translate T of a proper algebraic subgroup of G such that $W \subseteq V \cap T$ and

$$\dim W > \max\{0, \dim V + \dim T - \dim G\}.$$

Moreover, we denote by V^{oa} the complement in V of the union of its anomalous subvarieties.

In particular, for $G = \mathbb{G}_m^n$, V^{oa} is open in V by Theorem 1.4 of [BMZ07].

In this context, we have the following result of Habegger [Hab09b], which gives a uniform bound for the Weil height (see Chapter 2 for the definition of Weil height) of the points in V^{oa} that lie in a torsion coset of the right codimension (see also [Hab17] for an effective result).

Theorem 1.17 (Bounded Height conjecture)**.** *Let $V \subseteq \mathbb{G}_m^n$ be an irreducible subvariety of dimension d defined over $\overline{\mathbb{Q}}$. Then, $V^{oa} \cap (\mathbb{G}_m^n)^{[d]}$ is a set of bounded Weil height.*

Notice that if V is a curve, then the only possible anomalous subvariety is V itself, if V is contained in a translate of a proper algebraic subgroup. Thus, V^{oa} in this case is either empty or all of V, recovering Theorem 1 of [BMZ99].

Furthermore, Theorem 1.17 allows to prove the following, which provides an important partial result towards the Zilber–Pink conjecture for $\mathbb{G}_m^n$.

Theorem 1.18 ([Hab09b, Corollary 1.4])**.** *Let $V \subseteq \mathbb{G}_m^n$ be an irreducible subvariety defined over $\overline{\mathbb{Q}}$. Then, $V^{oa} \cap (\mathbb{G}_m^n)^{[\dim V+1]}$ is a finite set.*

Indeed, if $V^{oa} \neq \emptyset$, then

$$V \cap (\mathbb{G}_m^n)^{[\dim V+1]} \subseteq (V \setminus V^{oa}) \cup \left(V^{oa} \cap (\mathbb{G}_m^n)^{[\dim V+1]}\right).$$

As $V^{oa} \neq \emptyset$ is open by [BMZ07, Theorem 1.4], the right side is a proper closed subset of V, and therefore the Zilber–Pink conjecture holds for V.

1.3 Unlikely intersections in abelian varieties

We now turn our attention to abelian varieties, where the Zilber–Pink conjecture has seen substantial progress in recent years. As for multiplicative groups, recall that the special subvarieties of an abelian variety are precisely the translates of the abelian subvarieties by torsion points, again called torsion cosets. Recall also that for an abelian variety A and a non-negative integer n, we denote by $A^{[n]}$ the union of all special subvarieties of codimension at least n.

The abelian Bounded Height Conjecture, i.e. the analogue of Theorem 1.17 for abelian varieties, was established by Habegger in [Hab09a] (with a partial result for curves due to Rémond [Ré05, Lemme 3.3]). Building on another result by Rémond [Ré07, Corollaire 1.6], Habegger and Pila [HP16, Theorem 1.1] proved Conjecture 1.5 for curves in abelian varieties defined over $\overline{\mathbb{Q}}$. More recently, Barroero and Dill [BD22] extended this result to curves in abelian varieties defined over $\mathbb{C}$.

Theorem 1.19 (Habegger-Pila, Barroero-Dill)**.** *Let A be an abelian variety and $C \subseteq A$ an irreducible curve, both defined over $\mathbb{C}$. If C is not contained in a proper algebraic subgroup of A, then $C \cap A^{[2]}$ is a finite set.*

Previous partial results on the Zilber–Pink conjecture for curves defined over $\overline{\mathbb{Q}}$ in abelian varieties were obtained by various authors. Viada first proved finiteness in the case where C is not contained in the translate of a proper abelian subvariety and the abelian variety A is a power of an elliptic curve with complex multiplication [Via03]. This restriction on C was later removed in joint work with Rémond [RV03]. Ratazzi extended the result to the case where A is isogenous to a power of a simple abelian variety with complex multiplication [Rat08]. Carrizosa's lower bounds for the Néron–Tate height [Car08, Car09], combined with Rémond's upper bounds [Ré07], led to a proof of the conjecture for all abelian varieties with complex multiplication. Finally, the case of arbitrary powers was treated by Galateau and Viada [Gal10, Via08].

While the case of curves is now well understood, the study of higher dimensional subvarieties remains largely open. As for algebraic tori, the Zilber–Pink conjecture for hypersurfaces reduces to the Manin-Mumford conjecture (Theorem 1.3), proved for abelian varieties by Raynaud [Ray83a]. Furthermore, significant progress has been made for subvarieties of codimension 2, culminating in a recent result of Barroero and Dill, which follows from Corollary 1.6 of [BD22].

Theorem 1.20 (Barroero-Dill)**.** *Let $A/\mathbb{C}$ be an abelian variety of dimension n and $V \subseteq A$ an irreducible subvariety of dimension $n-2$, defined over $\mathbb{C}$ and not contained in any proper special subvariety of A. Then, $V \cap A^{[n-1]}$ is not Zariski dense in V.*

Partial results for codimension 2 subvarieties of abelian varieties defined over $\overline{\mathbb{Q}}$ had previously been proven by Checcoli, Veneziano and Viada [CVV14] for subvarieties in powers of elliptic curves with complex multiplication; by Hubschmid and Viada [HV19] in the non-CM case; and by Checcoli and Viada [CV14] for arbitrary products of CM elliptic curves.

As in the toric case, several important partial results are known for general subvarieties of abelian varieties. First, the openness of the non-anomalous locus was established by Rémond, who proved in [Ré09, Théorème 1.4] that V^{oa} is Zariski open in V.

Next, Habegger proved the following bounded height theorem, which may be viewed as the abelian analogue of Theorem 1.17.

Theorem 1.21 ([Hab09a])**.** *Let A be an abelian variety and $V \subseteq A$ an irreducible closed subvariety of dimension d, both defined over $\overline{\mathbb{Q}}$. Fix an ample symmetric line bundle on A and let $\widehat{h}$ be the associated Néron–Tate height on $A(\overline{\mathbb{Q}})$. Then $\widehat{h}$ is bounded on*

$$V^{oa} \cap A^{[d]}.$$

This result was subsequently used to prove the following finiteness result, analogous to Theorem 1.18, which constitutes an important partial case of the Zilber–Pink conjecture for abelian varieties.

Theorem 1.22 ([HP16, Theorem 9.15]). *Let A be an abelian variety and $V \subseteq A$ an irreducible closed subvariety of dimension d, both defined over $\overline{\mathbb{Q}}$. Then the set*

$$V^{oa} \cap A^{[d+1]}$$

is finite.

In particular, the same argument as in the toric case shows that the Zilber–Pink conjecture holds for any subvariety $V \subseteq A$ such that $V^{oa} \neq \emptyset$.

We conclude by mentioning an extension of Theorem 1.19 to the semiabelian setting, proved by Barroero, Kühne and Schmidt [BKS23], relying on the semiabelian Bounded Height conjecture proved by Kühne in [Kü20].

Theorem 1.23 (Barroero-Kühne-Schmidt). *Let G be a semiabelian variety and $C \subseteq G$ be an irreducible curve not contained in a proper algebraic subgroup of G, both defined over $\overline{\mathbb{Q}}$. Then $C \cap G^{[2]}$ is a finite set.*

This result extends to curves defined over $\mathbb{C}$ in semiabelian varieties defined over $\overline{\mathbb{Q}}$, by Theorem 14.1 of [BD24].

Remark 1.24. It can be shown, using an argument due to Zilber, that Theorem 1.19 implies Faltings's theorem (formerly known as the Mordell Conjecture, see Introduction). For a proof of this implication, see also [Cap23, Example 1.7].

1.3.1 Families of abelian varieties

So far, we have considered the case of fixed abelian varieties. However, all the conjectures and results presented above have natural analogues in the context of families of abelian varieties or, more formally, abelian schemes. This will be the setting of the problems studied in Chapters 3 and 4. Again, we assume that all the varieties are defined over $\mathbb{C}$, unless otherwise stated.

Let S be a regular, irreducible, quasi-projective variety and $\pi : \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$, i.e. a proper smooth group scheme such that for every $s \in S$ the fiber $\mathcal{A}_s := \pi^{-1}(s)$ is an abelian variety of dimension g.

Example 1.25. An important example of abelian scheme which will be used throughout Chapter 3 is the *Legendre family* of elliptic curves. Let $S = Y(2) := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and let $\lambda : \mathcal{E}_L \to S$ be the scheme with fibers given by

$$\mathcal{E}_\lambda : Y^2 Z = X(X - Z)(X - \lambda Z)$$

for each $\lambda \in S$. We will denote by $\mathcal{E}_L^n \to Y(2)$ the n-fold fibered power $\mathcal{E}_L \times_{Y(2)} \cdots \times_{Y(2)} \mathcal{E}_L$. Note that n sections $P_1, \ldots, P_n : Y(2) \to \mathcal{E}_L$ define a curve $(P_1, \ldots, P_n)(Y(2)) \subseteq \mathcal{E}_L^n$ which dominates the base $Y(2)$.

Let $\pi : \mathcal{A} \to S$ be an abelian scheme. A subgroup scheme $G \subseteq \mathcal{A}$ is a closed subvariety which contains the image of $G \times_S G$ under the addition morphism and the image of the zero section $O : S \to \mathcal{A}$, and is mapped to itself by the inversion morphism. A subgroup scheme $G \subseteq \mathcal{A}$ is called *flat* if $\pi|_G$ is flat. If S is a curve, this is equivalent (see [Har77, Proposition III.9.7]) to saying that every irreducible component of G dominates the base S. Note that, if G is a flat subgroup scheme, then for every $s \in S(\mathbb{C})$ the fiber G_s is an algebraic subgroup of $\mathcal{A}_s$. The dimension of this subgroup does not depend on s and it is usually called the relative dimension of G.

Remark 1.26. It is worth noting that certain abelian schemes, such as those appearing as universal families over Shimura varieties, can be realized as connected mixed Shimura varieties. A general construction is given in [Pin05a, Construction 2.9], while [BD24, Section 13] discusses the specific case of a fibered power of the Legendre family. In particular, if S is a curve and $\mathcal{A} \to S$ is the fibered power of a non-isotrivial elliptic scheme, then the special subvarieties of $\mathcal{A}$ are precisely the irreducible components of the flat subgroup schemes and the irreducible components of algebraic subgroups of the CM fibers; see [Hab13a, p. 144].

Let $\mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 2$, defined over $\mathbb{C}$. For each point $s \in S(\mathbb{C})$, denote by $\mathcal{A}_{s,\mathrm{tors}}$ the torsion subgroup of the fiber $\mathcal{A}_s$, and define

$$\mathcal{A}_{\mathrm{tors}} = \bigcup_{s \in S(\mathbb{C})} \mathcal{A}_{s,\mathrm{tors}}.$$

Observe that $\mathcal{A}_{\mathrm{tors}}$ coincides with the union of the kernels of the multiplication-by-N maps on $\mathcal{A}$, that is,

$$\mathcal{A}_{\mathrm{tors}} = \bigcup_{N \in \mathbb{Z} \setminus \{0\}} \ker[N],$$

where $[N] : \mathcal{A} \to \mathcal{A}$ denotes the multiplication-by-N morphism. Each torsion multisection $\ker[N]$ has codimension g in $\mathcal{A}$. Hence, for a subvariety $V \subseteq \mathcal{A}$ with $\dim V < g$, we expect the intersection $V \cap \ker[N]$ to be unlikely. This motivates the following result.

Theorem 1.27 (Relative Manin-Mumford conjecture)**.** *Let S be a regular, irreducible, quasi-projective variety and $\pi : \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$, both defined over $\mathbb{C}$. Let also $V \subseteq \mathcal{A}$ be an irreducible subvariety such that $\bigcup_{N \in \mathbb{Z}} [N]V$ is Zariski dense in $\mathcal{A}$. If $V(\mathbb{C}) \cap \mathcal{A}_{\mathrm{tors}}$ is Zariski dense in V, then $\dim V \geq g$.*

The Relative Manin–Mumford conjecture originated from ideas outlined by Zhang in his ICM talk [Zha98b], and was later formulated explicitly by Pink in [Pin05b, Conjecture 6.2]. It was recently proven by Gao and Habegger [GH23].

Earlier progress toward the conjecture had been made through a series of works by Masser and Zannier. They first established it for curves in abelian schemes of relative dimension 2, defined over $\mathbb{C}$ and isogenous to a product of elliptic schemes [MZ10, MZ12, MZ14], and later for more general abelian schemes of relative dimension 2 over bases defined over $\overline{\mathbb{Q}}$ [MZ15]. In collaboration with Corvaja, they also handled the case of relative dimension 2 over a base variety defined over $\mathbb{C}$ [CMZ18]. The case of curves in arbitrary abelian schemes defined over $\overline{\mathbb{Q}}$ was subsequently treated by Masser and Zannier [MZ20], while the surface case was addressed by Habegger [Hab13b] and by Corvaja, Tsimerman, and Zannier [CTZ23]. Finally, for general subvarieties of fibered products of elliptic schemes, the conjecture was proved by Kühne [Kü23].

In the setting of families of abelian varieties, Pink also proposed another conjecture. For an abelian scheme $\mathcal{A} \to S$, $s \in S(\mathbb{C})$ and an integer n let

$$\mathcal{A}_s^{[>n]} = \bigcup_{\operatorname{codim} H > n} H$$

where the union runs over all the algebraic subgroups H of the fiber $\mathcal{A}_s$. Define also

$$\mathcal{A}^{[>n]} = \bigcup_{s \in S(\mathbb{C})} \mathcal{A}_s^{[>n]}.$$

Conjecture 1.28 ([Pin05b, Conjecture 6.1]). *Let $\mathcal{A} \to S$ be an abelian scheme defined over $\mathbb{C}$ and $V \subseteq \mathcal{A}$ be an irreducible closed subvariety that is not contained in any proper closed subgroup scheme of $\mathcal{A}$, even after finite base changes. Then $V \cap \mathcal{A}^{[>\dim V]}$ is not Zariski dense in V.*

Remark 1.29. Conjecture 1.28 was originally formulated in the broader context of families of semiabelian varieties. However, this generalization was shown to be false by Bertrand [Ber11], who constructed explicit counterexamples based on the existence of so-called *Ribet sections* in certain semiabelian schemes. In subsequent work, Bertrand, Masser, Pillay, and Zannier [BMPZ16] proved that, for one-dimensional families of semiabelian surfaces of toric rank 1 defined over $\overline{\mathbb{Q}}$, Ribet sections are the only obstruction to the conjecture's validity. A complete and published account of Bertrand's original counterexample has recently been provided by Bertrand and Edixhoven [BE20].

Note that Conjecture 1.28 is weaker than Conjecture 1.5 for abelian schemes, as shown in [Pin05b, Theorem 6.3][2]. In the relative setting, partial results towards Conjecture 1.5 have been proved for curves in fibered powers of elliptic schemes.

Let S be a smooth, irreducible, quasi-projective curve and let $\mathcal{E} \to S$ be an elliptic scheme, both defined over $\overline{\mathbb{Q}}$. Assume that $\mathcal{E}$ is not isotrivial, i.e. it is not a constant family even after a base change. For $n \geq 2$, let $\pi : \mathcal{E}^n \to S$ be the n-fold fibered power of $\mathcal{E}$. Given a curve $C \subseteq \mathcal{E}^n$, each point $\mathbf{c} \in C(\mathbb{C})$ defines n points $P_1(\mathbf{c}), \dots, P_n(\mathbf{c})$ on the fiber $\mathcal{E}^n_{\pi(\mathbf{c})}$.

[2] This result was originally stated for families of semiabelian varieties, but in this context it is incorrect, as discussed earlier. However, the statement holds in the case of families of abelian varieties, as confirmed in Remark 5.4(4) of [BE20].

Theorem 1.30 (Barroero-Capuano [BC16]). *Let $C \subseteq \mathcal{E}^n$ be an irreducible curve defined over $\overline{\mathbb{Q}}$, not contained in a fixed fiber of $\mathcal{E}^n$ and such that the n points $P_1, \ldots, P_n$ defined by it are generically independent (i.e. no relation of the form $\sum_{i=1}^n a_i P_i = O$ with $a_i \in \mathbb{Z}$ not all zeroes holds identically). Then there are at most finitely many $\mathbf{c} \in C(\mathbb{C})$ such that there exist vectors $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{Z}^n$ that are linearly independent over $\mathbb{Q}$ and satisfying*

$$a_1 P_1(\mathbf{c}) + \ldots + a_n P_n(\mathbf{c}) = b_1 P_1(\mathbf{c}) + \ldots + b_n P_n(\mathbf{c}) = O.$$

This result can be equivalently reformulated by noting that, for fixed linearly independent vectors $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{Z}^n$, the set of points $(P_1, \ldots, P_n)$ of $\mathcal{E}^n \to S$ satisfying

$$a_1 P_1 + \ldots + a_n P_n = b_1 P_1 + \ldots + b_n P_n = O$$

defines a flat subgroup scheme of codimension 2 (see Lemma 2.2 of [BC16]). Conversely, every flat subgroup scheme is contained in a subgroup scheme defined by linear equations with integer coefficients of the same dimension [Hab13a, Lemma 2.5]. Thus, Theorem 1.30 is equivalent to stating that, if C is not contained in a proper flat subgroup scheme or in a fixed fiber, then the intersection of C with the union of all flat subgroup schemes of $\mathcal{E}^n$ of codimension at least 2 is finite.

Note also that for $n = 2$, the codimension 2 flat subgroup schemes are exactly the torsion multisections, so Theorem 1.30 reduces to the Relative Manin-Mumford conjecture, which was previously proved in this specific setting in the above-mentioned articles by Masser and Zannier [MZ10, MZ12, MZ14].

Since the intersection of a flat subgroup scheme with a fiber yields an algebraic subgroup of the same codimension in that fiber, Theorem 1.30 provides evidence towards Conjecture 1.28. However, it does not fully establish the conjecture in this setting, as it does not account for the algebraic subgroups of the fiber with non-trivial endomorphism ring. The following result addresses this aspect.

Theorem 1.31 (Barroero [Bar19]). *Let $C \subseteq \mathcal{E}^n$ be an irreducible curve defined over $\overline{\mathbb{Q}}$, not contained in a fixed fiber of $\mathcal{E}^n$ and such that the n points $P_1, \ldots, P_n$ defined by it are generically independent. Then there are at most finitely many $\mathbf{c} \in C(\mathbb{C})$ such that $\mathcal{E}_{\pi(\mathbf{c})}$ has complex multiplication and there exists $(a_1, \ldots, a_n) \in \mathrm{End}(\mathcal{E}_{\pi(\mathbf{c})})^n \setminus \{\mathbf{0}\}$ with*

$$a_1 P_1(\mathbf{c}) + \ldots + a_n P_n(\mathbf{c}) = O.$$

As before, Theorem 1.31 is equivalent to say that if C is not contained in a proper flat subgroup scheme or in a fixed fiber, then the intersection of C with the union of all proper algebraic subgroups of the CM fibers of $\mathcal{E}^n$ is finite.

Note that the conclusion of Theorem 1.31 is stronger than that of Conjecture 1.28. The conjecture considers only algebraic subgroups of codimension at least 2 in each fiber, whereas in the case where the fiber $\mathcal{E}_s$ has complex multiplication, an algebraic subgroup of codimension d in $\mathcal{E}_s^n$ corresponds to a special subvariety of codimension $d+1$ in $\mathcal{E}^n$. In particular, algebraic subgroups of codimension 1 in CM fibers are special subvarieties of codimension 2 in $\mathcal{E}^n$, which are not taken into account in Conjecture 1.28.

Observe also that, if $C \subseteq \mathcal{E}^n$ is contained in a fixed fiber or if $\mathcal{E}$ is isotrivial, then the analogues of Theorems 1.30 and 1.31 reduce to the case of a curve in a fixed abelian variety, which is already covered by Theorem 1.19. Therefore, by combining Theorems 1.19, 1.30, and 1.31, we obtain a proof of the Zilber–Pink conjecture for a curve in a fibered power of an elliptic scheme, when everything is defined over $\overline{\mathbb{Q}}$.

A natural extension of these results concerns products of powers of elliptic schemes. Let $\lambda : \mathcal{E}_\lambda \to Y(2)$ and $\mu : \mathcal{E}_\mu \to Y(2)$ be two copies of the Legendre scheme (we use subscripts to avoid ambiguity when dealing with fibered powers) and, for positive integers m, n, let $\mathcal{E}_\lambda^m \times \mathcal{E}_\mu^n \to Y(2) \times Y(2)$ be the product of fibered powers of these schemes.

Consider an irreducible curve $C \subseteq \mathcal{E}_\lambda^m \times \mathcal{E}_\mu^n$ defined over $\overline{\mathbb{Q}}$. As before, each point $\mathbf{c} \in C(\mathbb{C})$ defines m points $P_1(\mathbf{c}), \dots, P_m(\mathbf{c})$ on $\mathcal{E}_{\lambda(\mathbf{c})}$ and n points $Q_1(\mathbf{c}), \dots, Q_n(\mathbf{c})$ on $\mathcal{E}_{\mu(\mathbf{c})}$. Assume that the P_i are generically independent over $\mathrm{End}(\mathcal{E}_{\lambda|_C})$ and the same holds for the Q_j, which is equivalent to requiring that C is not contained in a proper flat subgroup scheme of $\mathcal{E}_\lambda^m \times \mathcal{E}_\mu^n \to Y(2) \times Y(2)$. Suppose also that $\mathcal{E}_\lambda$ and $\mathcal{E}_\mu$ are not generically isogenous when restricted to C.

Theorem 1.32 (Barroero-Capuano [BC17])**.** *Let $C \subseteq \mathcal{E}_\lambda^m \times \mathcal{E}_\mu^n$ as above. Then, there are at most finitely many $\mathbf{c} \in C(\mathbb{C})$ such that there exist vectors $(a_1, \dots, a_m) \in \mathrm{End}(\mathcal{E}_{\lambda|_C})^m \setminus \{\mathbf{0}\}$ and $(b_1, \dots, b_n) \in \mathrm{End}(\mathcal{E}_{\mu|_C})^n \setminus \{\mathbf{0}\}$ for which*

$$a_1 P_1(\mathbf{c}) + \ldots + a_m P_m(\mathbf{c}) = O_\lambda \qquad \text{and} \qquad b_1 Q_1(\mathbf{c}) + \ldots + b_n Q_n(\mathbf{c}) = O_\mu.$$

In combination with Theorems 1.19 and 1.30, this result implies that, in the product of two fibered powers of elliptic schemes under the above hypotheses, the intersection of a curve with the union of all flat subgroup schemes of codimension at least 2 is finite.

Together with Theorem 3.2, proved in Chapter 3, this addresses a large class of cases predicted by the Zilber–Pink conjecture for curves in the product of two powers of the Legendre family, with the exception of the case in which one factor has CM and there is a linear relation on the other factor, which will be treated in future work.

Finally, for general abelian schemes, we have the following result.

Theorem 1.33 (Barroero-Capuano [BC20])**.** *Let $\mathcal{A} \to S$ be an abelian scheme over a smooth irreducible curve S, and C an irreducible curve in A not contained in a proper subgroup scheme of $\mathcal{A}$, even after a finite base change. Suppose that $\mathcal{A}$, S and C are all defined over $\overline{\mathbb{Q}}$. Then, the intersection of C with the union of all flat subgroup schemes of $\mathcal{A}$ of codimension at least 2 is a* *finite set.*

We will continue the study of the Zilber–Pink conjecture in the setting of abelian schemes in Chapter 4, where we prove Theorem 4.1, a generalization of Theorem 1.31 to arbitrary abelian schemes.

Chapter 2

Heights

One fundamental tool in Diophantine geometry is the concept of a *height function*, which provides a measure of the “size” or “arithmetic complexity” of an algebraic point on a variety. One of the key properties we are interested in is that there should be at most finitely many points of bounded height and bounded degree.

We will construct height functions by first defining them on $\overline{\mathbb{Q}}$, then on projective spaces, and finally on projective varieties. We will also consider the important case of abelian varieties, where we can define a distinguished height function, called the *Néron-Tate* or *canonical* height, which satisfies particularly nice properties. In Section 2.6, we will establish a new explicit bound for the canonical height of the image of points under endomorphisms of an abelian variety.

The main references for this chapter are [BG06] and [HS13, Part B].

2.1 Absolute values and the product formula

2.1.1 Absolute values on general fields

Definition 2.1. Let K be a field. An absolute value is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ satisfying the following three properties:

1. for all $x \in K$, $|x| = 0$ if and only if $x = 0$;
2. for all $x, y \in K$, $|xy| = |x|\,|y|$;
3. for all $x, y \in K$, $|x + y| \leq |x| + |y|$.

If in addition we have $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in K$, we say that the absolute value is *nonarchimedean* or *ultrametric*. Otherwise, the absolute value is said to be *archimedean*.

Example 2.2. For any field K, we have a trivial (nonarchimedean) absolute value, defined by $|0| = 0$ and $|x| = 1$ for any $x \in K^{\times}$.

Another easy example is the usual absolute value on $\mathbb{R}$, $|x| = \max(x, -x)$, also denoted as $|x|_\infty$, which is archimedean.

Definition 2.3. Two absolute values $|\cdot|_1$ and $|\cdot|_2$ on K are *equivalent* if there exists a constant $\alpha > 0$ such that $|x|_1 = |x|_2^\alpha$ for every $x \in K$. Alternatively, $|\cdot|_1$ and $|\cdot|_2$ are equivalent if they define the same topology on K. An equivalence class of non-trivial absolute values on K is called a *place*.

One way of constructing absolute values is using valuations.

Definition 2.4. A valuation is a function $\nu : K \to \mathbb{R} \cup \{\infty\}$ satisfying the following properties:

1. for all $x \in K$, $\nu(x) = \infty$ if and only if $x = 0$;
2. for all $x, y \in K$, $\nu(xy) = \nu(x) + \nu(y)$;
3. for all $x, y \in K$, $\nu(x + y) \geq \min(\nu(x), \nu(y))$.

If ν is a valuation on K, we can define an absolute value on K by choosing $a \in \mathbb{R}_{>1}$ and setting

$$|x|_\nu = \begin{cases} a^{-\nu(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

In particular, this is a nonarchimedean absolute value and it is easy to see that its equivalence class does not depend on a.

2.1.2 Absolute values on number fields

Let p be a prime. For any $x \in \mathbb{Q}^\times$, there is a unique integer $\nu_p(x)$ such that

$$x = p^{\nu_p(x)} \cdot \frac{a}{b}$$

with a and b integers not divisible by p. Extending ν_p to 0 by setting $\nu_p(0) = \infty$ defines the so-called *p-adic valuation*. We will call the associated absolute value on $\mathbb{Q}$, $|x|_p = p^{-\nu_p(x)}$, the *p-adic absolute value*. It is clear that the p-adic absolute values and the usual absolute value are pairwise not equivalent and the following theorem (see [Cas86, Theorem 2.1] for the proof) shows that these are essentially the only absolute values on $\mathbb{Q}$.

Theorem 2.5 (Ostrowski, 1916)**.** *On $\mathbb{Q}$ any nontrivial absolute value is equivalent to either:*

- *The usual absolute value, $|\cdot|_\infty$, or*
- *A p-adic absolute value, $|\cdot|_p$, for some prime p.*

For a field K and a place v, we denote by K_v the completion of K with respect to any representative $|\cdot|_v$ of v.

Proposition 2.6 (Product formula)**.** *Let K be a number field. It is possible to choose a set M_K consisting of exactly one representative for each place of K such that, for every $x \in K^\times$, we have*

$$\sum_{v \in M_K} |x|_v^{d_v} = 1$$

where $d_v = [K_v : \mathbb{Q}_v]$.

In the case $K = \mathbb{Q}$, we can choose $M_\mathbb{Q}$ to consist of the usual absolute value $|\cdot|_\infty$ and the p-adic absolute values for every prime p. In this case, the product formula follows easily from the fundamental theorem of arithmetic.

For a general number field K, we can take M_K as the set of absolute values on K restricting to those in $M_\mathbb{Q}$ on $\mathbb{Q}$. However, these absolute values can also be explicitly described, as we now explain.

We start by describing archimedean absolute values. For each embedding $\sigma : K \hookrightarrow \mathbb{C}$, we define an archimedean absolute value by $|x|_\sigma = |\sigma(x)|_\infty$, where $|\cdot|_\infty$ denotes the usual archimedean absolute value on $\mathbb{C}$. Since $|\overline{z}|_\infty = |z|_\infty$ for every $z \in \mathbb{C}$, it follows that $|\cdot|_\sigma = |\cdot|_\tau$ whenever $\tau = \overline{\sigma}$.

In light of this, we list the $n := [K : \mathbb{Q}]$ embeddings $K \hookrightarrow \mathbb{C}$ as

$$\sigma_1, \ldots, \sigma_{r_1}, \sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}, \overline{\sigma_{r_1+1}}, \ldots, \overline{\sigma_{r_1+r_2}},$$

where $\sigma_1, \ldots, \sigma_{r_1}$ are the real embeddings and $\sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}$ are the complex ones, up to conjugation.

Thus, we get $r_1 + r_2$ non-trivial archimedean absolute values on K, $|\cdot|_{\sigma_1}, \ldots, |\cdot|_{\sigma_{r_1+r_2}}$, which are pairwise non-equivalent. One can also show that any non-trivial archimedean absolute value on K is equivalent to one of these; see for example [Wal00, Section 3.1.3]. Moreover, if v is an archimedean place of the number field K, then $K_v \cong \mathbb{R}$ if v corresponds to a real embedding, and $K_v \cong \mathbb{C}$ otherwise. Since $\mathbb{Q}_v \cong \mathbb{R}$ for every archimedean place, we have

$$d_v = [K_v : \mathbb{Q}_v] = \begin{cases} 1 & \text{if } v \text{ is real,} \\ 2 & \text{if } v \text{ is complex.} \end{cases}$$

Next, we describe the non-archimedean absolute values, which correspond to the non-zero prime ideals of $\mathcal{O}_K$.

To define the analogue of the p-adic valuation on K, let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$. The $\mathfrak{p}$-adic valuation $\nu_\mathfrak{p}$ is defined as follows. For any non-zero $x \in \mathcal{O}_K$, $\nu_\mathfrak{p}(x)$ is the unique integer such that there exists an ideal $\mathcal{I}$ of $\mathcal{O}_K$ satisfying

$$x\mathcal{O}_K = \mathfrak{p}^{\nu_\mathfrak{p}(x)} \cdot \mathcal{I},$$

where $\mathfrak{p}$ does not divide $\mathcal{I}$. Equivalently, $\nu_\mathfrak{p}(x)$ is the exponent of $\mathfrak{p}$ in the factorization of $x\mathcal{O}_K$ into prime ideals.

This definition extends to all of K by setting

$$\nu_\mathfrak{p}\left(\frac{a}{b}\right) = \nu_\mathfrak{p}(a) - \nu_\mathfrak{p}(b)$$

for any $a, b \in K^\times$, and we set $\nu_\mathfrak{p}(0) = \infty$ by definition.

Analogously to the definition of p-adic absolute values on $\mathbb{Q}$, we use this valuation to define the $\mathfrak{p}$-adic absolute value on K. For a prime ideal $\mathfrak{p}$ of K, let p be the prime number such that $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$.

We then define

$$|x|_\mathfrak{p} = p^{-\nu_\mathfrak{p}(x)/e(\mathfrak{p}|p)}$$

where $e(\mathfrak{p}|p) = \nu_\mathfrak{p}(p)$ is the ramification index of $\mathfrak{p}$ over p. The factor $e(\mathfrak{p}|p)$ ensures that $|p|_\mathfrak{p} = p^{-1}$, so that $|\cdot|_\mathfrak{p}$ extends $|\cdot|_p$. It can be verified that absolute values corresponding to different prime ideals are non equivalent and that any non-trivial non-archimedean absolute value on K is equivalent to one of these. Furthermore, if v is the non-archimedean place of K associated with $\mathfrak{p}$, we have $d_v = e(\mathfrak{p}|p)f(\mathfrak{p}|p)$, where $f(\mathfrak{p}|p) = [\mathcal{O}_K/\mathfrak{p} : \mathbb{F}_p]$ is the inertia degree of $\mathfrak{p}$. As before, we also have that $d_v = [K_v : \mathbb{Q}_v]$.

The following theorem generalizes Theorem 2.5 by showing that these are the only non-trivial absolute values up to equivalence. For the proof we refer to Ostrowski's original articles [Ost16] and [Ost35], or [Con, Theorem 3.3].

Theorem 2.7 (Ostrowski)**.** *Every non-trivial absolute value on a number field K is equivalent to one of the absolute values described above, namely, either a $\mathfrak{p}$-adic absolute value associated with a unique non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, or an Archimedean absolute value induced by a real or complex embedding of K.*

From now on, we will denote by M_K the set of absolute values of K described above, so that the product formula holds (see [HS13, Proposition B.1.2] for a proof). We will write M_K^∞ for the set of archimedian (or infinite) places, and M_K^0 for the set of non-archimedian (or finite) places.

2.2 The Weil height on $\overline{\mathbb{Q}}$

In light of our discussion on absolute values and the product formula, we now define the Weil height on $\overline{\mathbb{Q}}$.

Definition 2.8. Let $\alpha \in \overline{\mathbb{Q}}$ be an algebraic number. The *absolute logarithmic Weil height* of α is defined by

$$h(\alpha) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} d_v \log \max\{1, |\alpha|_v\},$$

where K is any number field containing α. We also define the *absolute multiplicative Weil height* of α as $H(\alpha) = \exp(h(\alpha))$.

First, observe that the sum is always finite, since $|\alpha|_v = 1$ for all but finitely many $v \in M_K$. Notice also that $h(\alpha)$ does not depend on the choice of K.

Example 2.9. If $\alpha = \frac{a}{b} \in \mathbb{Q}$ is a rational number, where a, b are coprime integers, then we have

$$\prod_{p \text{ prime}} \max\left\{1, \left|\frac{a}{b}\right|_p\right\} = |b|_\infty$$

and

$$\max\left\{1, \left|\frac{a}{b}\right|_\infty\right\} = \begin{cases} |\frac{a}{b}|_\infty & \text{if } |a|_\infty > |b|_\infty \\ 1 & \text{otherwise} \end{cases}$$

so that

$$h\left(\frac{a}{b}\right) = \log(|b|_\infty) + \log\max\left\{1, \left|\frac{a}{b}\right|_\infty\right\} = \log\max\left\{|a|_\infty, |b|_\infty\right\}.$$

We now present several properties of the Weil height.

Proposition 2.10.

1. *For any $\alpha \in \overline{\mathbb{Q}}$, $h(\alpha) \geq 0$. We have $h(\alpha) = 0$ if and only if $\alpha = 0$ or α is a root of unity (Kronecker).*

2. *For any $\alpha, \beta \in \overline{\mathbb{Q}}$, $h(\alpha\beta) \leq h(\alpha) + h(\beta)$. Moreover, if β is a root of unity, then $h(\alpha\beta) = h(\alpha)$.*

3. *For any $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$, we have $h(\alpha_1 + \ldots + \alpha_n) \leq h(\alpha_1) + \ldots + h(\alpha_n) + \log n$.*

4. *For any non-zero $\alpha \in \overline{\mathbb{Q}}$ and any $n \in \mathbb{Z}$, $h(\alpha^n) = |n| \cdot h(\alpha)$.*

5. *For any $\alpha \in \overline{\mathbb{Q}}$ and any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $h(\sigma(\alpha)) = h(\alpha)$.*

Observe that it is not possible to replace $\log n$ in part 3 with any smaller constant, as demonstrated by the example $\alpha_1 = \ldots = \alpha_n = 1$. For the proofs of these properties see Section 1.5 of [BG06].

Proposition 2.11 (Proposition 3.2 of [Zan14])**.** *Let $R(x) = \frac{P(x)}{Q(x)} \in \overline{\mathbb{Q}}(x)$ be a rational function, with $P(x), Q(x) \in \overline{\mathbb{Q}}[x]$ coprime polynomials. Then, for every $\alpha \in \overline{\mathbb{Q}}$ such that $\mathbb{Q}(\alpha) \neq 0$, we have $h(R(\alpha)) = \deg(R)h(\alpha) + O(1)$, where $\deg(R) = \max\{\deg(P), \deg(Q)\}$ and the bounded function $O(1)$ depends only on R.*

The following finiteness theorem is one of the reasons why height functions are so widely used in Diophantine geometry, serving as a foundational result for many finiteness results, such as those described in Chapters 3 and 4.

Theorem 2.12 (Northcott)**.** *Let B and D be real numbers. Then, the set*

$$\left\{\alpha \in \overline{\mathbb{Q}} : h(\alpha) \leq B \text{ and } [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq D\right\}$$

is finite.

Remark 2.13. Let $\alpha \in \overline{\mathbb{Q}}$ and let $f(x) = a_d x^d + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$ be its minimal polynomial. Assume that $\gcd(a_d, \ldots, a_0) = 1$ and that $a_d \neq 0$. Denote by $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$ the (distinct) complex roots of f. Then we can give an alternative definition of the Weil height of α as follows:

$$h(\alpha) = \frac{1}{d}\left(\log|a_d| + \sum_{j=1}^{d} \log\max\left\{1, |\alpha_j|_\infty\right\}\right).$$

This allows us to compute $h(\alpha)$ without computing the absolute values.

2.3 The Weil height on $\mathbb{P}^n(\overline{\mathbb{Q}})$

We now begin to extend the Weil height to a more geometric setting.

Definition 2.14. Let $P = [x_0 : \ldots : x_n] \in \mathbb{P}^n(\overline{\mathbb{Q}})$ and let K be a number field containing $x_0, \ldots, x_n$. The absolute logarithmic Weil height of P is defined as:

$$h(P) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} d_v \log\max\{|x_0|_v, \ldots, |x_n|_v\}.$$

As before, we define the absolute multiplicative Weil height of P as $H(P) = \exp(h(P))$.

As for the Weil height on $\overline{\mathbb{Q}}$, this definition does not depend on the choice of K. Furthermore, by the product formula, the height also does not depend on the choice of the homogeneous coordinates of P, ensuring that it is well-defined. Moreover, it satisfies $h(\sigma(P)) = h(P)$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Remark 2.15. For $n = 1$, we recover the previous definition of the height on $\overline{\mathbb{Q}}$, since we can embed $\overline{\mathbb{Q}}$ into $\mathbb{P}^1(\overline{\mathbb{Q}})$ by $\alpha \mapsto [1 : \alpha]$.

More generally, we can embed $\overline{\mathbb{Q}}^n$ into $\mathbb{P}^n(\overline{\mathbb{Q}})$ by

$$\begin{aligned} \overline{\mathbb{Q}}^n &\longrightarrow \mathbb{P}^n(\overline{\mathbb{Q}}) \\ (\alpha_1, \ldots, \alpha_n) &\longmapsto [1 : \alpha_1 : \ldots : \alpha_n]. \end{aligned}$$

Thus, we can define the height of a point $(\alpha_1, \ldots, \alpha_n) \in \overline{\mathbb{Q}}^n$ as

$$h((\alpha_1, \ldots, \alpha_n)) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} d_v \log \max\{1, |\alpha_1|_v, \ldots, |\alpha_n|_v\},$$

where we can choose $K = \mathbb{Q}(a_1, \ldots, \alpha_n)$, as the definition does not depend on the choice of K.

For a point $P = [x_0 : \ldots : x_n] \in \mathbb{P}^n(\overline{\mathbb{Q}})$, we define its field of definition as

$$\mathbb{Q}(P) = \mathbb{Q}\left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right)$$

for any index j such that $x_j \neq 0$. In particular, up to permutations and normalization of the coordinates, we may assume that $x_0 = 1$. From this, it follows that $h(P) \geq 0$ and, if $x_0 = 1$, $h(P) \geq h(x_i)$ for every $i = 1, \ldots, n$. Thus, Theorem 2.12 implies the following generalization of Northcott's theorem to $\mathbb{P}^n(\overline{\mathbb{Q}})$.

Theorem 2.16. *Let B and D be real numbers. Then, the set*

$$\left\{P \in \mathbb{P}^n(\overline{\mathbb{Q}}) : h(P) \leq B \text{ and } [\mathbb{Q}(P) : \mathbb{Q}] \leq D\right\}$$

is finite.

In particular, this implies that for any fixed number field K, the set

$$\{P \in \mathbb{P}^n(K) : h(P) \leq T\}$$

is finite for every $T \geq 0$.

Finally, we generalize Kronecker's theorem (part 1 of Proposition 2.10) to projective spaces.

Theorem 2.17 (Kronecker)**.** *Let $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$, and assume that $x_0 = 1$, as above. Then $h(P) = 0$ if and only if, for every $j = 1, \ldots, n$, $x_j = 0$ or x_j is a root of unity.*

2.3.1 Heights of matrices

Let $M = (m_{i,j}) \in \mathrm{Mat}_n(\overline{\mathbb{Q}})$. We associate to M two natural heights:

- the *affine height*, defined by

$$H_{\mathrm{aff}}(M) = \prod_{v \in M_K} \max\left\{1, \max_{1 \leq i,j \leq n} \{|m_{i,j}|_v\}\right\}^{\frac{d_v}{[K:\mathbb{Q}]}}$$

  where K is a number field containing all the entries of M. This coincides with the absolute multiplicative Weil height of M regarded as a point of $\overline{\mathbb{Q}}^{n^2}$;

- the *entry-wise height*, defined by

$$H_{\max}(M) = \max_{1\leq i,j\leq n} \{H(m_{i,j})\}.$$

The affine and entry-wise heights enjoy many useful properties with respect to usual matrix operations, which we now collect.

Proposition 2.18. *Let $A, B \in \mathrm{Mat}_n(\overline{\mathbb{Q}})$. Then:*

1. $H_{\max}(A) \leq H_{\text{aff}}(A) \leq H_{\max}(A)^{n^2}$;
2. $H_{\max}(A+B) \leq 2H_{\max}(A)H_{\max}(B)$;
3. $H_{\max}(AB) \leq nH_{\max}(A)^n H_{\max}(B)^n$;
4. $H(\det(A)) \leq n! \cdot H_{\text{aff}}(A)^n$;
5. *if A is invertible,* $H_{\max}(A^{-1}) \leq n! \cdot (n-1)! \cdot H_{\text{aff}}(A)^{2n-1}$.

Proof. Let $A = (a_{i,j})$ and $B = (b_{i,j})$ and fix a number field K containing all entries of A and B.

1. Since $\max\{1, |a_{i,j}|_v\} \leq \max\left\{1, \max_{1\leq i,j\leq n}\{|a_{i,j}|_v\}\right\}$, we clearly have

$$H(a_{i,j}) = \prod_{v\in M_K} \max\{1, |a_{i,j}|_v\}^{\frac{d_v}{[K:\mathbb{Q}]}} \leq H_{\text{aff}}(A)$$

which implies that $H_{\max}(A) \leq H_{\text{aff}}(A)$. Moreover, recall that

$$\max\left\{1, \max_{1\leq i,j\leq n}\{|a_{i,j}|_v\}\right\} \leq \prod_{1\leq i,j\leq n} \max\{1, |a_{i,j}|_v\}$$

which implies that $H_{\text{aff}}(A) \leq \prod_{1\leq i,j\leq n} H(a_{i,j}) \leq H_{\max}(A)^{n^2}$.

2. The claim follows from the inequality

$$H(a_{i,j} + b_{i,j}) \leq 2H(a_{i,j})H(b_{i,j}) \leq 2H_{\max}(A)H_{\max}(B),$$

which is a direct consequence of part (3) of Proposition 2.10.

3. Let $AB = (c_{i,j})$, where $c_{i,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j}$. Then, applying Proposition 2.10 yields

$$H(c_{i,j}) \leq n \cdot \prod_{k=1}^{n} H(a_{i,k})H(b_{k,j}) \leq nH_{\max}(A)^n H_{\max}(B)^n$$

which implies $H_{\max}(AB) \leq nH_{\max}(A)^n H_{\max}(B)^n$.

4. Recall that
$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$
where S_n denotes the symmetric group on n elements and $\operatorname{sgn}(\sigma) \in \{\pm 1\}$ is the sign of the permutation σ. Hence $\det(A)$ is the sum of $n!$ monomials of degree n in the entries of A. In particular, for every place $v \in M_K$, we have
$$|\det(A)|_v \leq \begin{cases} n! \cdot \max\limits_{1 \leq i,j \leq n} \{|a_{i,j}|_v\}^n & \text{if } v \text{ is archimedean} \\ \max\limits_{1 \leq i,j \leq n} \{|a_{i,j}|_v\}^n & \text{if } v \text{ is non-archimedean} \end{cases}$$
Hence,
$$\begin{aligned} \prod_{v \in M_K^0} \max\{1, |\det(A)|_v\}^{\frac{d_v}{[K:\mathbb{Q}]}} &\leq \prod_{v \in M_K^0} \left(\max\left\{1, \max_{1 \leq i,j \leq n} \{|a_{i,j}|_v\}\right\}^n \right)^{\frac{d_v}{[K:\mathbb{Q}]}} \\ &= \left(\prod_{v \in M_K^0} \max\left\{1, \max_{1 \leq i,j \leq n} \{|a_{i,j}|_v\}\right\}^{\frac{d_v}{[K:\mathbb{Q}]}} \right)^n \end{aligned}$$
and
$$\begin{aligned} \prod_{v \in M_K^\infty} \max\{1, |\det(A)|_v\}^{\frac{d_v}{[K:\mathbb{Q}]}} &\leq \prod_{v \in M_K^\infty} \left(n! \max\left\{1, \max_{1 \leq i,j \leq n} \{|a_{i,j}|_v\}\right\}^n \right)^{\frac{d_v}{[K:\mathbb{Q}]}} \\ &= (n!)^{\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K^\infty} d_v} \left(\prod_{v \in M_K^\infty} \max\left\{1, \max_{1 \leq i,j \leq n} \{|a_{i,j}|_v\}\right\}^{\frac{d_v}{[K:\mathbb{Q}]}} \right)^n \\ &= n! \left(\prod_{v \in M_K^\infty} \max\left\{1, \max_{1 \leq i,j \leq n} \{|a_{i,j}|_v\}\right\}^{\frac{d_v}{[K:\mathbb{Q}]}} \right)^n \end{aligned}$$
since $\sum_{v \in M_K^\infty} d_v = [K : \mathbb{Q}]$. So, we have
$$\begin{aligned} H(\det(A)) &= \prod_{v \in M_K} \max\{1, |\det(A)|_v\}^{\frac{d_v}{[K:\mathbb{Q}]}} \\ &= \prod_{v \in M_K^\infty} \max\{1, |\det(A)|_v\}^{\frac{d_v}{[K:\mathbb{Q}]}} \cdot \prod_{v \in M_K^0} \max\{1, |\det(A)|_v\}^{\frac{d_v}{[K:\mathbb{Q}]}} \\ &\leq n! \cdot H_{\text{aff}}(A)^n. \end{aligned}$$

5. The case $n = 1$ is trivial, so assume $n \geq 2$. Recall that $A^{-1} = \frac{1}{\det(A)} \cdot C^t$, where $C = ((-1)^{i+j}\mu_{i,j})$ is the cofactor matrix and $\mu_{i,j}$ is the (i,j)-minor[1] of A. Then, by part (4), $H((-1)^{i+j}\mu_{i,j}) \leq (n-1)! \cdot H_{\text{aff}}(A)^{n-1}$, so that $H_{\max}(C) \leq (n-1)! \cdot H_{\text{aff}}(A)^{n-1}$. Therefore, $H_{\max}(A^{-1}) \leq H(\det(A)) \cdot H_{\max}(C) \leq n! \cdot (n-1)! \cdot H_{\text{aff}}(A)^{2n-1}$. □

[1]Some authors use the word *minor* to denote just the matrix obtained from A by removing a row and a column. In this case, by minor we mean the determinant of such a submatrix.

2.4 The Weil height on Projective varieties

After studying the Weil height on $\mathbb{P}^n(\overline{\mathbb{Q}})$, we now extend this notion to the $\overline{\mathbb{Q}}$-points of arbitrary projective varieties. This extension is provided by the *Weil height machine*, which associates height functions to divisors on projective varieties, allowing us to measure the arithmetic complexity of points on more general varieties.

In general, if a projective variety V, defined over a number field, admits an embedding into $\mathbb{P}^n$, we can define the height of a point in $V(\overline{\mathbb{Q}})$ by viewing it as a point in $\mathbb{P}^n(\overline{\mathbb{Q}})$.

As a first example, we consider $\mathbb{P}^m \times \mathbb{P}^n$. In this case, we have the Segre embedding

$$\begin{aligned} S_{m,n} : \mathbb{P}^m \times \mathbb{P}^n &\hookrightarrow \mathbb{P}^N \\ (\mathbf{x}, \mathbf{y}) &\longmapsto [x_0y_0 : x_0y_1 : \ldots : x_iy_j : \ldots : x_my_n] \end{aligned}$$

where $N = (m+1)(n+1) - 1$. This allows us to define $h_{\mathbb{P}^m \times \mathbb{P}^n}(\mathbf{x}, \mathbf{y}) := h_{\mathbb{P}^N}(S_{m,n}(\mathbf{x}, \mathbf{y}))$, which satisfies the following property.

Proposition 2.19. *For every $\mathbf{x} \in \mathbb{P}^m(\overline{\mathbb{Q}})$ and $\mathbf{y} \in \mathbb{P}^n(\overline{\mathbb{Q}})$, we have*

$$h(S_{m,n}(\mathbf{x}, \mathbf{y})) = h(\mathbf{x}) + h(\mathbf{y}).$$

More generally, to extend this approach to arbitrary projective varieties, we can use any morphism into a projective space.

Definition 2.20. Let V be a projective variety defined over $\overline{\mathbb{Q}}$ and let $\phi : V \to \mathbb{P}^n$ be a morphism defined over $\overline{\mathbb{Q}}$. The height on V relative to ϕ is defined as

$$h_\phi(P) := h(\phi(P)) \qquad \text{for any } P \in V(\overline{\mathbb{Q}})$$

where h is the absolute logarithmic Weil height on $\mathbb{P}^n$ defined before.

Since height functions are defined via morphisms into projective space, one may worry about their dependence on the choice of such morphisms. The next proposition ensures that if two morphisms come from the same complete linear system, the associated height functions differ by a bounded function.

Proposition 2.21. *Let V be a projective variety defined over $\overline{\mathbb{Q}}$. Let $H \subseteq \mathbb{P}^n$ and $H' \subseteq \mathbb{P}^m$ be hyperplanes. Let $\phi : V \to \mathbb{P}^n$ and $\psi : V \to \mathbb{P}^m$ be morphisms such that ϕ^*H and ψ^*H' are linearly equivalent. Then we have*

$$h_\phi(P) = h_\psi(P) + O(1)$$

for every $P \in V(\overline{\mathbb{Q}})$. The bounded function $O(1)$ depends on V, ϕ and ψ but not on P.

While the height functions we introduced depend on a choice of morphism into projective space, a more intrinsic approach exists. Instead of relying on embeddings, one can define heights directly in terms of divisors on the variety. This construction, due to Weil, is also known as *Weil's height machine* and it allows us to translate geometric relations into arithmetical statements about heights.

Theorem 2.22 (Weil's height machine)**.** *Let V be a smooth projective variety defined over $\overline{\mathbb{Q}}$. Then, there exists a map*

$$\begin{aligned} \mathrm{Div}(V) &\longrightarrow \{\textit{functions } V(\overline{\mathbb{Q}}) \to \mathbb{R}\} \\ D \quad &\longmapsto h_{V,D} \end{aligned}$$

which is, up to a bounded function $O(1)$, uniquely defined by the following three properties:

1. *(Normalization) Let $H \subseteq \mathbb{P}^n$ be an hyperplane. Then, for every $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$, we have*

$$h_{\mathbb{P}^n,H}(P) = h(P) + O(1),$$

where h is the absolute logarithmic height on $\mathbb{P}^n(\overline{\mathbb{Q}})$.

2. *(Functoriality) Let $\phi : V \to W$ be a morphism and let $D \in \mathrm{Div}(W)$ be a divisor. Then, for every $P \in V(\overline{\mathbb{Q}})$, we have*

$$h_{V,\phi^* D}(P) = h_{W,D}(\phi(P)) + O(1).$$

3. *(Additivity) Let $D, D' \in \mathrm{Div}(V)$. Then, for every $P \in V(\overline{\mathbb{Q}})$, we have*

$$h_{V,D+D'}(P) = h_{V,D}(P) + h_{V,D'}(P) + O(1).$$

Moreover, it satisfies the following additional properties:

4. *(Linear equivalence) If $D, D' \in \mathrm{Div}(V)$ are two linearly equivalent divisors, then*

$$h_{V,D}(P) = h_{V,D'}(P) + O(1)$$

for every $P \in V(\overline{\mathbb{Q}})$.

5. *(Positivity) Let $D \in \mathrm{Div}(V)$ be an effective divisor, and let B be the set of base points of the associated linear system $|D|$. Then, there is a constant $\gamma \in \mathbb{R}$ such that, for every $P \in (V \setminus B)(\overline{\mathbb{Q}})$, we have*

$$h_{V,D}(P) \geq \gamma.$$

6. *(Northcott) Let $D \in \mathrm{Div}(V)$ be an ample divisor. Then, for any constants $B, C \in \mathbb{R}$ the set*

$$\{P \in V(\overline{\mathbb{Q}}) : [\mathbb{Q}(P) : \mathbb{Q}] \leq B \textit{ and } h_{V,D}(P) \leq C\}$$

is finite.

The bounded functions $O(1)$ and the constant in property 5, depend on the varieties, divisors and morphisms, but not on the points on the variety.

The main idea behind this construction is to first define $h_{V,D}$ when D is very ample (or even base point free). If D is very ample, then the map associated to its complete linear system $\phi_{|D|} : V \to \mathbb{P}^n$ is an embedding such that $\phi_{|D|}^* H$ is linearly equivalent to D for every hyperplane $H \subseteq \mathbb{P}^n$. Then, we define $h_{V,D} = h \circ \phi_{|D|}$. Proposition 2.21 implies that, up to a bounded function, this definition does not depend on the chosen embedding. In particular, if $H \subseteq \mathbb{P}^n$ is an hyperplane, the corresponding embedding $\phi_{|H|} : \mathbb{P}^n \to \mathbb{P}^n$ is the identity, so property 1 follows easily.

For a general divisor $D \in \mathrm{Div}(V)$, it is known (see [HS13, Theorem A.3.2.3]) that there are two very ample divisors $D_1, D_2 \in \mathrm{Div}(V)$, such that $D = D_1 - D_2$. Thus, we define

$$h_{V,D}(P) = h_{V,D_1}(P) - h_{V,D_2}(P)$$

for every $P \in V(\overline{\mathbb{Q}})$.

For the proof of Theorem 2.22, we refer to the proof of Theorem B.3.2 of [HS13].

Remark 2.23. If the variety V is not smooth, we can use either Cartier divisors or line bundles instead of Weil divisors. See for example Theorem B.3.6 of [HS13].

Example 2.24. Let C be a smooth projective curve of genus 0 defined over a number field k. Then, the anticanonical divisor $-K_C$ (which we assume defined over k, too) is very ample and the image of C by the associated embedding is a conic in $\mathbb{P}^2$ defined over k. In particular, the height $h_{C,-K_C}$ corresponds to the restriction of the usual height on $\mathbb{P}^2$ to the image of this embedding.

Let C be a smooth projective curve of genus 1 defined over a number field k and let $O \in C(k)$. Then, the divisor $3O$ is very ample and the associated embedding (defined over k) is

$$\begin{aligned} \phi_{|3O|} : \quad C &\hookrightarrow \quad \mathbb{P}^2 \\ P \neq O &\longmapsto [x(P) : y(P) : 1] \\ O &\longmapsto \quad [0 : 1 : 0] \end{aligned}$$

where $x, y \in k(C)$ are two functions satisfying

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

for some constants $a_1, a_2, a_3, a_4, a_6 \in k$. In other words, C is isomorphic to the elliptic curve E in $\mathbb{P}^2$ defined by the equation above and with identity element $[0 : 1 : 0]$. Thus, the height on C associated to $3O$ is just the logarithmic height on $\mathbb{P}^2$ restricted to E.

2.5 Canonical height functions

The height functions obtained from Theorem 2.22 are well-defined only up to an additive bounded term. In this section, we introduce a refinement that selects a distinguished representative within this equivalence class, known as the canonical height. In particular, we will consider the special case of abelian varieties, in which the canonical heights satisfy additional properties related to the group structure.

Theorem 2.25 (Néron, Tate)**.** *Let V be a smooth projective variety defined over a number field and $D \in \mathrm{Div}(V)$. Let $\phi : V \to V$ be a morphism such that $\phi^* D \sim \alpha D$ for some $\alpha > 1$. Then there exists a unique function $\widehat{h}_{V,\phi,D} : V(\overline{\mathbb{Q}}) \to \mathbb{R}$, which we call the* canonical height *on V with respect to ϕ and D, such that*

i) $\widehat{h}_{V,\phi,D}(P) = h_{V,D}(P) + O(1)$ *for every* $P \in V(\overline{\mathbb{Q}})$.

ii) $\widehat{h}_{V,\phi,D}(\phi(P)) = \alpha \cdot \widehat{h}_{V,\phi,D}(P)$ *for every* $P \in V(\overline{\mathbb{Q}})$.

iii) If $D' \in \mathrm{Div}(V)$ is linearly equivalent to D, then $\widehat{h}_{V,\phi,D} = \widehat{h}_{V,\phi,D'}$.

Moreover, $\widehat{h}_{V,\phi,D}$ can be computed as follows:

$$\widehat{h}_{V,\phi,D}(P) = \lim_{n\to\infty} \frac{1}{\alpha^n} h_{V,D}(\phi^n(P))$$

where $\phi^n = \phi \circ \ldots \circ \phi$ is the n-th iterate of ϕ.

To illustrate the construction of canonical heights, we consider the special case of projective spaces.

Example 2.26. Let $d \geq 2$ and consider the morphism

$$\begin{array}{rccc} \pi_d : & \mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ & [x_0 : \ldots : x_n] & \longmapsto & \left[x_0^d : \ldots : x_n^d\right]. \end{array}$$

If $H \subset \mathbb{P}^n$ is any hyperplane, then $\pi_d^* H \sim dH$. Thus, we can define a canonical height $\widehat{h}_{\mathbb{P}^n,\pi_d,H}$. Since $h_{\mathbb{P}^n,H}(P) = h(P) + O(1)$ and $h(\pi_d(P)) = dh(P)$ for every $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$, we have

$$\widehat{h}_{\mathbb{P}^n,\pi_d,H}(P) = \lim_{N\to\infty} \frac{1}{d^N} h_{\mathbb{P}^n,H}(\pi_d^N(P)) = \lim_{N\to\infty} \frac{1}{d^N} \left(d^N h(P) + O(1)\right) = h(P).$$

Thus, in this case, the canonical height coincides with the absolute logarithmic Weil height on $\mathbb{P}^n$.

Proposition 2.27. *Let V be a smooth projective variety defined over a number field and $D \in \mathrm{Div}(V)$ be an ample divisor. Let $\phi : V \to V$ be a morphism such that $\phi^* D \sim \alpha D$ for some*

$\alpha > 1$ and let $\widehat{h}_{V,\phi,D}$ be the associated canonical height defined in the theorem above. Then $\widehat{h}_{V,\phi,D}(P) \geq 0$ for every $P \in V(\overline{\mathbb{Q}})$ and $\widehat{h}_{V,\phi,D}(P) = 0$ if and only if the set

$$\left\{P, \phi(P), \phi^2(P), \ldots, \phi^n(P), \ldots\right\}$$

is finite.

2.5.1 Preliminaries on abelian varieties

In this section, we collect the foundational definitions and results concerning abelian varieties that will serve as a basis for our development of canonical heights and for the rest of the thesis. It is not intended as a comprehensive treatment of abelian varieties, for which we refer the reader to [BL04, Mil08, Mum08].

Recall that an abelian variety is a projective, connected and geometrically reduced group variety, i.e. a projective, connected and geometrically reduced variety A with a base point $O \in A$ and morphisms

$$\mu : A \times A \to A$$
$$\iota : A \to A$$

which endow A with the structure of a group. In other words, these maps satisfy the following identities for all $P, Q, R \in A$:

$$\mu(P, O) = \mu(O, P) = P$$
$$\mu(P, \iota(P)) = \mu(\iota(P), P) = O$$
$$\mu(\mu(P, Q), R) = \mu(P, \mu(Q, R))$$

We say that the abelian variety is defined over K if the variety A, along with the morphisms μ and ι, is defined over K and $O \in A(K)$. It is a classical fact that abelian varieties are smooth, and moreover, the group law on A is commutative. Therefore, we write $\mu(P, Q) = P + Q$ and $\iota(P) = -P$.

For any integer n, let $[n]_A : A \to A$ be the multiplication-by-n map. When no ambiguity arises, we will write $[n]$ instead of $[n]_A$.

Proposition 2.28 (Mumford's formula)**.** *Let A be an abelian variety, $D \in \mathrm{Div}(A)$ and $n \in \mathbb{Z}$. Then*

$$[n]^*D \sim \frac{n^2+n}{2} D + \frac{n^2-n}{2} [-1]^*D.$$

In particular, if D is symmetric*, i.e. $[-1]^*D \sim D$, then $[n]^*D \sim n^2 D$. On the other hand, if D is* antisymmetric*, i.e. $[-1]^*D \sim -D$, then $[n]^*D \sim nD$.*

In this thesis we will be mainly working with abelian varieties defined over $\mathbb{C}$, which we will identify with their set of complex points. It is well-known that if A is an abelian variety of dimension g defined over $\mathbb{C}$, then $A(\mathbb{C})$ is a complex torus, i.e. $A(\mathbb{C}) \cong V/\Lambda$ for some g-dimensional $\mathbb{C}$-vector space V and some lattice $\Lambda \subseteq V$. After fixing bases of V and Λ, we have that $\Lambda = \Pi\mathbb{Z}^{2g}$, for some matrix $\Pi \in \mathrm{Mat}_{g\times 2g}(\mathbb{C})$ called *period matrix*.

Let A, B be two abelian varieties. A *homomorphism* is a morphism $f : A \to B$ of group varieties (in other words, it is a morphism of algebraic varieties which is also a group homomorphism). When $B = A$ such a map is called an *endomorphism*. A homomorphism $f : A \to B$ is called an *isogeny* if it is surjective and it has finite kernel.

We denote by $\mathrm{Hom}(A, B)$ the set of homomorphisms from A to B and we define $\mathrm{End}(A) := \mathrm{Hom}(A, A)$ to be the set of all endomorphisms. Moreover, we define

$$\mathrm{Hom}^0(A, B) := \mathrm{Hom}(A, B) \otimes \mathbb{Q} \qquad \mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbb{Q}.$$

Note that $\mathrm{Hom}(A, B)$ is an abelian group under point-wise addition and, similarly, $\mathrm{End}(A)$ is a ring where the multiplication is given by composition of maps. We will always assume that all the morphisms are defined over an algebraic closure of the ground field.

Given an endomorphism f of $A = V/\Lambda$, by Proposition 1.2.1 of [BL04], there is a unique linear map $F : V \to V$ with $F(\Lambda) \subseteq \Lambda$ and inducing f on A. The restriction F_Λ of F to Λ is $\mathbb{Z}$-linear and completely determines both F and f.

Fix bases of V and Λ, and let Π be the corresponding period matrix, i.e. the matrix representing the basis of Λ in terms of the basis of V. With respect to these bases, F and F_Λ are given by matrices $\rho_a(f) \in \mathrm{Mat}_g(\mathbb{C})$ and $\rho_r(f) \in \mathrm{Mat}_{2g}(\mathbb{Z})$, respectively. Since $F(\Lambda) \subseteq \Lambda$, we must have

$$\rho_a(f) \cdot \Pi = \Pi \cdot \rho_r(f). \tag{2.1}$$

The associations $F \mapsto \rho_a(f)$ and $F_\Lambda \mapsto \rho_r(f)$ extend to injective ring homomorphisms

$$\rho_a : \mathrm{End}^0(A) \longrightarrow \mathrm{Mat}_g(\mathbb{C})$$
$$\rho_r : \mathrm{End}^0(A) \longrightarrow \mathrm{Mat}_{2g}(\mathbb{Q})$$

called the analytic representation and the rational representation of $\mathrm{End}^0(A)$, respectively.

We denote by $\widehat{A} = \mathrm{Pic}^0(A)$ the dual abelian variety, i.e. the group of line bundles on A that are algebraically equivalent to zero. Given a point $x \in A$, we denote by T_x the translation-by-x map. If L is an arbitrary line bundle on A, we have a homomorphism

$$\begin{aligned} \Phi_L : A &\longrightarrow \widehat{A} \\ x &\longmapsto T_x^* L \otimes L^{-1} \end{aligned} \tag{2.2}$$

and we call $K(L)$ its kernel. A *polarization* is an isogeny $A \to \widehat{A}$ of the form Φ_L for some ample line bundle L. We say that a polarization is *principal* if it is an isomorphism (i.e. $\deg \Phi_L = 1$). Recall that any two algebraically equivalent ample line bundles on A define the same polarization.

We denote by $\chi(L)$ the Euler characteristic of L.

To any polarization Φ_L on A corresponds a positive definite Hermitian form $H_L = c_1(L) : V \times V \to \mathbb{C}$, given by the first Chern class of the line bundle L. It is worth noting that in the literature, the term polarization may refer either to the ample line bundle L (up to algebraic equivalence), the associated isogeny Φ_L, or the Hermitian form H_L. These notions are equivalent; see, for example, Section 4.1 of [BL04]. We denote by $E_L = \operatorname{Im}(H_L)$ the alternating Riemann form associated with L, which takes integer values on the lattice Λ.

Given an ample line bundle L on A, there exists a basis of Λ, called *symplectic basis*, such that the alternating Riemann form $E_L : \Lambda \times \Lambda \to \mathbb{Z}$ is represented by the matrix

$$\begin{pmatrix} 0 & \mathbf{D} \\ -\mathbf{D} & 0 \end{pmatrix}$$

where $\mathbf{D} := \operatorname{diag}(d_1, \dots, d_g)$ is a diagonal matrix, with $d_1, \dots, d_g$ positive integers such that d_i divides d_{i+1} for each $i = 1, \dots, g-1$. We call $\mathbf{D}$ the *type* of the polarization Φ_L and we define the *Pfaffian* of E_L as $\operatorname{Pf}(E_L) = \det(\mathbf{D})$ [BL04, Section 3.2]. The degree of the isogeny Φ_L is called the *degree* of the polarization and it is easy to prove that it is equal to $\operatorname{Pf}(E_L)^2 = \det(E_L)$.

Next, we define the Rosati (anti-)involution on $\operatorname{End}^0(A)$ with respect to the polarization Φ_L as:

$$\begin{aligned} {}^\dagger : \operatorname{End}^0(A) &\longrightarrow \operatorname{End}^0(A) \\ f &\longmapsto f^\dagger = \Phi_L^{-1} \circ \widehat{f} \circ \Phi_L \end{aligned} \tag{2.3}$$

where $\widehat{f} \in \operatorname{End}^0(\widehat{A})$ denotes the dual of f and, with a slight abuse of notation, we also denote by Φ_L the corresponding element of $\operatorname{Hom}^0(A, \widehat{A})$. This map is $\mathbb{Q}$-linear and satisfies $(fg)^\dagger = g^\dagger f^\dagger$ for all $f, g \in \operatorname{End}^0(A)$. In particular, if Φ_L is a principal polarization, the Rosati involution restricts to an involution on $\operatorname{End}(A)$.

2.5.2 Canonical heights on abelian varieties

Having introduced the general theory, we now explore canonical heights in the context of abelian varieties. In this setting, the group structure imposes additional constraints on height functions, leading to stronger arithmetic properties. In particular, by Proposition 2.28, Theorem 2.25 implies that we can construct a canonical height associated with either a symmetric or an antisymmetric divisor. We will first consider the symmetric case, which is the primary focus of the next section as well as Chapters 3 and 4.

Theorem 2.29. *Let A be an abelian variety defined over a number field, and let D be a symmetric divisor on A. Then there is a unique function $\widehat{h}_{A,D} : A(\overline{\mathbb{Q}}) \to \mathbb{R}$, called* canonical height on A relative to D, *satisfying the following properties:*

i) $\widehat{h}_{A,D}(P) = h_{A,D}(P) + O(1)$ *for every* $P \in A(\overline{\mathbb{Q}})$.

ii) For every $n \in \mathbb{Z}$, $\widehat{h}_{A,D}([n]P) = n^2 \cdot \widehat{h}_{A,D}(P)$ *for every* $P \in A(\overline{\mathbb{Q}})$.

iii) If $D' \in \mathrm{Div}(A)$ *is symmetric, then* $\widehat{h}_{A,D+D'} = \widehat{h}_{A,D} + \widehat{h}_{A,D'}$.

iv) If $D' \in \mathrm{Div}(A)$ *is linearly equivalent to* D, *then* $\widehat{h}_{A,D} = \widehat{h}_{A,D'}$.

v) (Parallelogram Law) $\widehat{h}_{A,D}(P + Q) + \widehat{h}_{A,D}(P - Q) = 2\widehat{h}_{A,D}(P) + 2\widehat{h}_{A,D}(Q)$, *for all* $P, Q \in A(\overline{\mathbb{Q}})$.

Classically, $\widehat{h}_{A,D}$ is defined as the canonical height on A with respect to $[2] : A \to A$, using Theorem 2.25:

$$\widehat{h}_{A,D}(P) = \lim_{n\to\infty} \frac{1}{4^n} h_{A,D}([2^n]P).$$

However, one can show that replacing 2 with any integer $m \neq -1, 0, 1$ yields the same canonical height on A.

Remark 2.30. If D is an ample and symmetric divisor on A, then Proposition 2.27 implies that $\widehat{h}_{A,D}(P) \geq 0$ for every $P \in A(\overline{\mathbb{Q}})$. Moreover, $\widehat{h}_{A,D}(P) = 0$ if and only if the set

$$\{P, [2]P, [4]P, \ldots, [2^n]P, \ldots\}$$

is finite. This finiteness condition is in turn equivalent to the existence of integers $0 \leq i < j$ such that $[2^i]P = [2^j]P$, which is equivalent to P having finite order, i.e. , P being a torsion point.

Furthermore, if D is a symmetric and nef divisor on A, then for any ample and symmetric divisor H and any $n > 0$, the divisor $nD + H$ is ample and symmetric, as follows from Kleiman's criterion, and we have

$$n\widehat{h}_{A,D} = \widehat{h}_{A,nD+H} - \widehat{h}_{A,H} \geq -\widehat{h}_{A,H}.$$

Since H is ample, we have $\widehat{h}_{A,H} \geq 0$, and as $n > 0$ is arbitrary, it follows that $\widehat{h}_{A,D} \geq 0$ as well. However, as shown in [KS16], when D is nef and symmetric, the set

$$\left\{P \in A(\overline{\mathbb{Q}}) : \widehat{h}_{A,D} = 0\right\}$$

may contain more than just the torsion points.

We can now state an analogous theorem for antisymmetric divisors.

Theorem 2.31. *Let A be an abelian variety defined over a number field, and let D be an antisymmetric divisor on A. Then there is a unique canonical height function $\widehat{h}_{A,D} : A(\overline{\mathbb{Q}}) \to \mathbb{R}$, such that:*

i) $\widehat{h}_{A,D}(P) = h_{A,D}(P) + O(1)$ *for every* $P \in A(\overline{\mathbb{Q}})$.

ii) If $D' \in \mathrm{Div}(A)$ *is antisymmetric, then* $\widehat{h}_{A,D+D'} = \widehat{h}_{A,D} + \widehat{h}_{A,D'}$.

iii) If $D' \in \mathrm{Div}(A)$ *is linearly equivalent to* D*, then* $\widehat{h}_{A,D} = \widehat{h}_{A,D'}$.

iv) For all $P, Q \in A(\overline{\mathbb{Q}})$,

$$\widehat{h}_{A,D}(P+Q) = \widehat{h}_{A,D}(P) + \widehat{h}_{A,D}(Q).$$

In particular, $\widehat{h}_{A,D}([n]P) = n \cdot \widehat{h}_{A,D}(P)$ *for every* $P \in A(\overline{\mathbb{Q}})$ *and any* $n \in \mathbb{Z}$.

Similar to the symmetric case, the assumption that D is antisymmetric implies that $[2]^*D \sim 2D$. Thus, we can apply Theorem 2.25 to define the canonical height $\widehat{h}_{A,D}$ as:

$$\widehat{h}_{A,D}(P) = \lim_{n\to\infty} \frac{1}{2^n} h_{A,D}([2^n]P).$$

Now, we combine Theorems 2.29 and 2.31 to extend the definition of canonical heights to arbitrary divisors.

Theorem 2.32. *Let A be an abelian variety defined over a number field, and let D be a divisor on A. Then, there is a unique function $\widehat{h}_{A,D} : A(\overline{\mathbb{Q}}) \to \mathbb{R}$ such that $\widehat{h}_{A,D} = h_{A,D} + O(1)$ and $\widehat{h}_{A,D}(O) = 0$, called* canonical height on A relative to D*, satisfying the following properties:*

i) If $D' \in \mathrm{Div}(A)$ *is linearly equivalent to* D*, then* $\widehat{h}_{A,D} = \widehat{h}_{A,D'}$.

ii) If $D, D' \in \mathrm{Div}(A)$*, then* $\widehat{h}_{A,D+D'} = \widehat{h}_{A,D} + \widehat{h}_{A,D'}$.

iii) Let B be another abelian variety (also defined over a number field), and let $\phi : B \to A$ be a morphism. Then

$$\widehat{h}_{B,\phi^*D} = \widehat{h}_{A,D} \circ \phi - \widehat{h}_{A,D}(\phi(O_B)).$$

In particular, if ϕ is a homomorphism, $\widehat{h}_{B,\phi^*D} = \widehat{h}_{A,D} \circ \phi$.

Given a divisor D on A, the key idea in constructing the associated canonical height is to define the divisors $D^+ = D + [-1]^*D$ and $D^- = D - [-1]^*D$. Then, D^+ is symmetric and D^- is antisymmetric, so we can define $\widehat{h}_{A,D^+}$ and $\widehat{h}_{A,D^-}$ using the theorems above. Hence, since $D^+ + D^- = 2D$, we can define

$$\widehat{h}_{A,D} = \frac{\widehat{h}_{A,D^+} + \widehat{h}_{A,D^-}}{2}.$$

2.6 Canonical height bounds for endomorphisms of abelian varieties

Let A be an abelian variety of dimension g defined over $\overline{\mathbb{Q}}$, and let D be a symmetric divisor on A. Since D is symmetric, Theorem 2.29 states that

$$\widehat{h}_{A,D}([n]P) = n^2 \cdot \widehat{h}_{A,D}(P)$$

for any $P \in A(\overline{\mathbb{Q}})$ and for every $n \in \mathbb{Z}$. The aim of this section is to generalize this identity to arbitrary endomorphisms of A.

It was noted by Naumann [Nau04] that, if $\mathrm{End}^0(A)$ is $\mathbb{Q}$, an imaginary quadratic field or a definite quaternion algebra over $\mathbb{Q}$, and if D is an ample symmetric divisor, then

$$\widehat{h}_{A,D}(f(P)) = (\deg f)^{1/g} \cdot \widehat{h}_{A,D}(P)$$

for any $f \in \mathrm{End}(A)$ and any $P \in A(\overline{\mathbb{Q}})$, recovering a well known fact for elliptic curves.

In general, however, we cannot expect an identity of this form, as illustrated by the following example. Consider $A = E \times E$, where E is any elliptic curve with identity element O, and let $D = (O \times E) + (E \times O)$. Define the endomorphism $f : A \to A$ by

$$f(P_1, P_2) = (P_1, 2P_2).$$

Since we can write $D = \pi_1^*(O) + \pi_2^*(O)$, where π_1 and π_2 are the projections onto the two factors, we obtain

$$\widehat{h}_{A,D}(P_1, P_2) = \widehat{h}_{E,O}(\pi_1(P_1, P_2)) + \widehat{h}_{E,O}(\pi_2(P_1, P_2)) = \widehat{h}_{E,O}(P_1) + \widehat{h}_{E,O}(P_2)$$

by Theorem 2.32. Choosing either $P_1 = O$ or $P_2 = O$, we conclude that there is no constant γ such that

$$\widehat{h}_{A,D}(f(P)) = \widehat{h}_{E,O}(P_1) + 4\widehat{h}_{E,O}(P_2) = \gamma \cdot \left(\widehat{h}_{E,O}(P_1) + \widehat{h}_{E,O}(P_2)\right) = \gamma \cdot \widehat{h}_{A,D}(P)$$

for every $P = (P_1, P_2) \in A(\overline{\mathbb{Q}})$. Nonetheless, since the divisor O is ample, it follows that

$$\widehat{h}_{A,D}(P) \leq \widehat{h}_{A,D}(f(P)) \leq 4\widehat{h}_{A,D}(P).$$

More generally, if D is ample and symmetric, there exist constants $0 \leq \gamma_1 \leq \gamma_2$ such that

$$\gamma_1 \cdot \widehat{h}_{A,D}(P) \leq \widehat{h}_{A,D}(f(P)) \leq \gamma_2 \cdot \widehat{h}_{A,D}(P).$$

In particular, γ_1 must be taken equal to 0 if f is not an isogeny, while it can be chosen strictly positive if f is an isogeny.

To prove the upper bound, recall that since D is ample, there exists an integer N_2 such that $nD - f^*D$ is ample for all $n \geq N_2$, see for instance [Laz04, Example 1.2.10]. This implies

$$N_2 \cdot \widehat{h}_{A,D}(P) - \widehat{h}_{A,D}(f(P)) = \widehat{h}_{A,N_2D-f^*D}(P) \geq 0$$

giving the upper bound with $\gamma_2 = N_2$.

For the lower bound, first observe that if f is not finite, then the dimension of $\ker(f)$ is positive and, in particular, there is a non-torsion point $P \in A(\overline{\mathbb{Q}})$ for which $f(P) = O$. Therefore, we must have $\gamma_1 = 0$ in this case. On the other hand, if f is finite then f^*D is ample. Thus, as before, there exists an integer $N_1 > 0$ such that $nf^*D - D$ is ample for any $n \geq N_1$. This means that

$$N_1 \cdot \widehat{h}_{A,D}(f(P)) - \widehat{h}_{A,D}(P) = \widehat{h}_{A,N_1f^*D-D}(P) \geq 0$$

from which we deduce the lower bound, with $\gamma_1 = \frac{1}{N_1} > 0$.

If f is an isogeny, the existence of these bounds also follows from Theorem B in [Lee16].

Unfortunately, this method does not provide effective values for γ_1 and γ_2, although explicit computations may be possible for specific choices of D and f.

The main result of this section is the following theorem, which gives explicit values for γ_1 and γ_2 in terms of the eigenvalues of the analytic representation of $f^\dagger f$, where † is the Rosati involution defined by the polarization associated to D. Define

$$\alpha_D^-(f) = \min\{\alpha_1, \ldots, \alpha_g\} \quad \text{and} \quad \alpha_D^+(f) = \max\{\alpha_1, \ldots, \alpha_g\},$$

where $\alpha_1, \ldots, \alpha_g$ are the eigenvalues (counted with multiplicities) of $\rho_a(f^\dagger f)$. We will prove in Lemma 2.34 that these eigenvalues are real and non-negative.

Theorem 2.33. *Let A be an abelian variety defined over $\overline{\mathbb{Q}}$, and let D be an ample symmetric divisor on A. Then, for every endomorphism $f : A \to A$, we have*

$$\alpha_D^-(f) \cdot \widehat{h}_{A,D}(P) \leq \widehat{h}_{A,D}(f(P)) \leq \alpha_D^+(f) \cdot \widehat{h}_{A,D}(P)$$

for every $P \in A(\overline{\mathbb{Q}})$. Moreover, these constants are the best possible, meaning that we cannot replace $\alpha_D^+(f)$ and $\alpha_D^-(f)$ with a smaller and a larger constant, respectively.

We will prove this theorem in Section 2.6.2.

2.6.1 Properties of endomorphisms and line bundles of abelian varieties

Fix an ample divisor D on $A = \mathbb{C}^g/\Lambda$ and let $L = \mathcal{O}_A(D)$ be the associated line bundle. In the following, † denotes the Rosati involution induced by the polarization Φ_L corresponding to L.

We start with a classical result about the eigenvalues of $\rho_a(f^\dagger f)$. The following proof is inspired by an argument by Masser and Wüstholz [MW94] (used also in Section 4.4).

Lemma 2.34. *Let $f \in \mathrm{End}^0(A)$. Then all the eigenvalues of $\rho_a(f^\dagger f)$ are real and non-negative. If $f \neq 0$, then at least one eigenvalue is positive.*

Proof. By Proposition 5.1.1 of [BL04], we have that $H_L(\rho_a(f)v, w) = H_L(v, \rho_a(f^\dagger)w)$, for every $v, w \in \mathbb{C}^g$, where $H_L : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$ is the Hermitian form associated with the ample line bundle L. Thus, if $\mathcal{H}_L$ is the matrix representing H_L, we have

$$\rho_a(f^\dagger) = \mathcal{H}_L^{-1}\overline{\rho_a(f)}^t\mathcal{H}_L$$

where $\overline{M}^t$ is the conjugate transpose of the matrix M.

Since L is ample, H_L is positive definite, and therefore there is an invertible matrix S such that $\mathcal{H}_L = \overline{S}^t S$. Thus, we have

$$\rho_a(f^\dagger f) = \mathcal{H}_L^{-1} \cdot \overline{\rho_a(f)}^t \cdot \mathcal{H}_L \cdot \rho_a(f) = S^{-1}(\overline{S}^t)^{-1}\overline{\rho_a(f)}^t\overline{S}^t S\rho_a(f).$$

By setting $X = S \cdot \rho_a(f) \cdot S^{-1}$, we have that

$$\rho_a(f^\dagger f) = S^{-1}(\overline{S}^t)^{-1} \cdot \overline{\rho_a(f)}^t \cdot \overline{S}^t S \cdot \rho_a(f) \cdot S^{-1}S = S^{-1}\overline{X}^t XS$$

proving that $\rho_a(f^\dagger f)$ has non-negative real eigenvalues, since $\overline{X}^t X$ is a positive semidefinite matrix and eigenvalues are invariant under change of basis. In particular, as Hermitian matrices are diagonalizable, this also implies that $\overline{X}^t X$ cannot have all zero eigenvalues unless it is the zero matrix. However, if X has entries $x_{i,j} \in \mathbb{C}$ and $\overline{X}^t X = \mathbf{0}$, then $0 = \mathrm{tr}(\overline{X}^t X) = \sum_{i,j=1}^g |x_{i,j}|^2$, which implies that $X = S \cdot \rho_a(f) \cdot S^{-1} = \mathbf{0}$ and thus $\rho_a(f) = \mathbf{0}$. □

Notice that for $f \in \mathrm{End}(A)$, the matrix $\rho_a(f^\dagger f)$ has only positive eigenvalues if and only if X is invertible, which is the case precisely when $\rho_a(f)$ is invertible, i.e. when f is an isogeny.

Denote by $P^a_{f^\dagger f}(x)$ and $P^r_{f^\dagger f}(x)$ the characteristic polynomial of $f^\dagger f$ with respect to the analytic and the rational representations, respectively. Using [BL04, Proposition 5.1.2] and the previous lemma, $P^a_{f^\dagger f}$ and $P^r_{f^\dagger f}$ are real polynomials and we have

$$P^r_{f^\dagger f}(x) = \left(P^a_{f^\dagger f}(x)\right)^2. \tag{2.4}$$

With these notations, we have the following generalization of Lemma 2.1 of [Lan88] (see also [BL04, Proposition 5.1.6]).

Lemma 2.35. *Let L be an ample line bundle, $f \in \mathrm{End}(A)$ and $a, b \in \mathbb{Z}$, with $b > 0$. Then,*

$$\chi(f^*L^{-b} \otimes L^a) = \chi(L) \cdot b^g \cdot P^a_{f^\dagger f}\left(\frac{a}{b}\right).$$

Proof. Fix $b > 0$ an integer. By Corollary 3.6.2 of [BL04], we have

$$\chi(f^*L^{-b} \otimes L^a)^2 = \deg(\Phi_{f^*L^{-b} \otimes L^a})$$

where the map Φ_L was defined in (2.2). By [BL04, Corollary 2.4.6] we have

$$\Phi_{f^*L^{-b} \otimes L^a} = -[b]\Phi_{f^*L} + [a]\Phi_L \quad \text{and} \quad \Phi_{f^*L} = \widehat{f}\Phi_L f = \Phi_L f^\dagger f.$$

Then, recalling that for every $\varphi \in \mathrm{End}(A)$, $\deg(\varphi) = \det(\rho_r(\varphi))$ [BL04, eq. (1.2)], we get

$$\begin{aligned}
\chi(f^*L^{-b} \otimes L^a)^2 &= \deg\left(-[b]\Phi_L f^\dagger f + [a]\Phi_L\right) \\
&= \deg \Phi_L \cdot \deg\left(-[b]f^\dagger f + [a]\right) \\
&= \deg \Phi_L \cdot \det\left(\rho_r\left(-[b] \cdot f^\dagger f + [a]\right)\right) \\
&= \deg \Phi_L \cdot \det\left(-b \cdot \rho_r(f^\dagger f) + a \cdot \mathbf{1}_{2g}\right) \\
&= \deg \Phi_L \cdot b^{2g} \cdot \det\left(-\rho_r(f^\dagger f) + \frac{a}{b} \cdot \mathbf{1}_{2g}\right) \\
&= \chi(L)^2 \cdot b^{2g} \cdot P^r_{f^\dagger f}\left(\frac{a}{b}\right) = \chi(L)^2 \cdot b^{2g} \cdot \left(P^a_{f^\dagger f}\left(\frac{a}{b}\right)\right)^2
\end{aligned}$$

by Equation (2.4). Here $\mathbf{1}_{2g}$ is the $2g \times 2g$ identity matrix. It follows that

$$\chi(f^*L^{-b} \otimes L^a) = \pm\chi(L) \cdot b^g \cdot P^a_{f^\dagger f}\left(\frac{a}{b}\right).$$

Fix $b > 0$ arbitrary. Since L is ample, we have $\chi(L) > 0$. Moreover, for all sufficiently large $a > 0$, the divisor $f^*L^{-b} \otimes L^a$ is ample by Kleinman's criterion, hence $\chi(f^*L^{-b} \otimes L^a) > 0$. Finally, since $P^a_{f^\dagger f}$ is a monic polynomial (see [BL04, after proof of Proposition 5.1.2]), $P^a_{f^\dagger f}(\frac{a}{b})$ is also positive for all sufficiently large $a > 0$, completing the proof. □

For the reader's convenience, we also recall the following theorem, which combines results by Kempf [Kem, Theorem 2] and by Mumford [Mum08, Section 16]. Here, given a line bundle M on A, we denote by $H^i(A, M)$ the i-th cohomology group of M. Recall also that we denote by $K(M)$ the kernel of the homomorphism $\Phi_M : A \to \widehat{A}$.

Theorem 2.36. *Let M and M' be line bundles on an abelian variety A of dimension g, with M*

ample. Consider the polynomial $P_{M,M'}(x) \in \mathbb{Q}[x]$ (of degree g) such that

$$P_{M,M'}(n) = \chi(M^n \otimes M')$$

for every $n \in \mathbb{Z}$. Then:

(i) All roots of $P_{M,M'}$ are real and $\dim K(M')$ is equal to the multiplicity of 0 as a root,

(ii) (Mumford's vanishing theorem) If $K(M')$ is finite, there is a unique integer $i = i(M')$, with $0 \leq i(M') \leq g$, such that $H^k(A, M') = 0$ for $k \neq i$ and $H^i(A, M') \neq 0$. Moreover, $K(M'^{-1})$ is finite[2] and $i(M'^{-1}) = g - i(M')$.

(iii) Counting roots with multiplicities, assume that $P_{M,M'}$ has N_- negative roots and N_+ positive roots, then:

$$H^k(A, M') = 0, \quad \text{if } 0 \leq k < N_+$$
$$H^{g-k}(A, M') = 0, \quad \text{if } 0 \leq k < N_-.$$

Finally, we have the following characterization of ample line bundles.

Proposition 2.37. *[BL04, Proposition 4.5.2] A line bundle M on A is ample if and only if $K(M)$ is finite and $H^0(A, M) \neq 0$.*

2.6.2 Proof of Theorem 2.33

Given an abelian variety A of dimension g defined over a number field, an ample symmetric divisor D and $f \in \mathrm{End}(A)$, let $\alpha_1, \ldots, \alpha_g$ be the eigenvalues (counted with multiplicities) of $\rho_a(f^\dagger f)$, where the Rosati involution is defined with respect to the polarization $L = \mathcal{O}_A(D)$.

Define $\alpha_D^-(f) = \min\{\alpha_1, \ldots, \alpha_g\}$ and $\alpha_D^+(f) = \max\{\alpha_1, \ldots, \alpha_g\}$, as before. Notice that, by Lemma 2.34, $\alpha_D^-(f)$ is non-negative and it is positive if and only if f is surjective, which is compatible with what we said in the introduction. Moreover, $\alpha_D^+(f) > 0$ for every $f \neq 0$.

Proof of Theorem 2.33. The claim is trivially true for $f = 0$, so we will assume that $f \neq 0$ for the rest of the proof. Let $\lambda = \frac{a}{b}$ be a rational number, with $b > 0$, and let L be the line bundle associated to D. As above, consider L as a polarization on A and define the Rosati involution with respect to this line bundle.

[2]This follows from [BL04, Lemma 2.4.7 (c)].

We start by proving the upper bound. Consider the line bundle $M = f^*L^{-b} \otimes L^a$. Then, for every $n \in \mathbb{Z}$,

$$P_{L,M}(n) = \chi(L^n \otimes M) = \chi(f^*L^{-b} \otimes L^{n+a}) = \chi(L) \cdot b^g \cdot P^a_{f^\dagger f}\left(\frac{n+a}{b}\right)$$

by Lemma 2.35. Thus, we have

$$\begin{aligned} P_{L,M}(x) &= \chi(L) \cdot b^g \cdot P^a_{f^\dagger f}\left(\frac{x+a}{b}\right) \\ &= \chi(L) \cdot b^g \cdot \prod_{i=1}^{g}\left(\frac{x+a}{b} - \alpha_i\right) \\ &= \chi(L) \cdot \prod_{i=1}^{g}\left(x - (b\alpha_i - a)\right). \end{aligned}$$

Combining Proposition 2.37 and Theorem 2.36, we obtain that M is ample if and only if all the roots of $P_{L,M}$ are negative, which is equivalent to say that $\frac{a}{b} > \alpha_i$ for every $i = 1, \ldots, g$.

This implies that if $\lambda = \frac{a}{b} > \alpha_D^+(f)$, then the divisor $aD - bf^*D$ is ample and symmetric and therefore, by Remark 2.30 and Theorem 2.32, we have

$$a \cdot \widehat{h}_{A,D}(P) - b \cdot \widehat{h}_{A,D}(f(P)) = a \cdot \widehat{h}_{A,D}(P) - b \cdot \widehat{h}_{A,f^*D}(P) = \widehat{h}_{A,aD-bf^*D}(P) \geq 0$$

for every $P \in A(\overline{\mathbb{Q}})$, which is equivalent to $\widehat{h}_{A,D}(f(P)) \leq \lambda \cdot \widehat{h}_{A,D}(P)$. Since this is true for every $\lambda \in \mathbb{Q}$ such that $\lambda > \alpha_D^+(f)$, this implies that $\widehat{h}_{A,D}(f(P)) \leq \alpha_D^+(f) \cdot \widehat{h}_{A,D}(P)$.

In order to prove the lower bound, we consider the line bundle $M = f^*L^b \otimes L^{-a}$. By Theorem 2.36 and Proposition 2.37, M is ample if and only if $K(M)$ is finite and $H^g(A, M^{-1}) \neq 0$. Using Lemma 2.35 as before, we get that

$$P_{L,M^{-1}}(x) = \chi(L) \cdot b^g \cdot P^a_{f^\dagger f}\left(\frac{x+a}{b}\right) = \chi(L) \cdot \prod_{i=1}^{g}\left(x - (b\alpha_i - a)\right).$$

By [BL04, Lemma 2.4.7(c)], $K(M) = K(M^{-1})$, and Theorem 2.36 implies that $K(M^{-1})$ is finite and $H^g(A, M^{-1}) \neq 0$ if and only if all the roots of $P_{L,M^{-1}}$ are positive, that is, if and only if $\frac{a}{b} < \alpha_i$ for every $i = 1, \ldots, g$.

Again, this means that for every $\lambda = \frac{a}{b} < \alpha_D^-(f)$, the divisor $bf^*D - aD$ is ample and symmetric and thus we have

$$b \cdot \widehat{h}_{A,D}(f(P)) - a \cdot \widehat{h}_{A,D}(P) = b \cdot \widehat{h}_{A,f^*D}(P) - a \cdot \widehat{h}_{A,D}(P) = \widehat{h}_{A,bf^*D-aD}(P) \geq 0$$

for every $P \in A(\overline{\mathbb{Q}})$, which is equivalent to $\widehat{h}_{A,D}(f(P)) \geq \lambda \cdot \widehat{h}_{A,D}(P)$. Since this is true for every $\lambda \in \mathbb{Q}$ such that $\lambda < \alpha_D^-(f)$, this implies that $\widehat{h}_{A,D}(f(P)) \geq \alpha_D^-(f) \cdot \widehat{h}_{A,D}(P)$.

We now prove that the constants $\alpha_D^-(f), \alpha_D^+(f)$ are optimal.

Consider the $\mathbb{Q}$-divisor $\lambda D - f^*D$. Observe that the proof above shows that $\lambda D - f^*D$ is ample if and only if $\lambda > \alpha_D^+(f)$. From this we deduce that, if $\lambda \in \mathbb{Q}$ and $\lambda < \alpha_D^+(f)$, then $\lambda D - f^*D$ is not nef. Otherwise, $(\lambda + \varepsilon)D - f^*D$ would be ample for every $\varepsilon > 0$ [Laz04, Corollary 1.4.10], which is impossible for ε small enough.

Then, assume that $0 \leq \widetilde{\alpha} < \alpha_D^+(f)$ is such that $\widehat{h}_{A,D}(f(P)) \leq \widetilde{\alpha} \cdot \widehat{h}_{A,D}(P)$ for every $P \in A(\overline{\mathbb{Q}})$. Without loss of generality we can assume that $\widetilde{\alpha}$ is rational. Then, since D is ample, f^*D is nef and, thus, $f^*D + D$ is ample. Therefore, we have that

$$\widehat{h}_{A,f^*D+D}(P) \leq \widehat{h}_{A,(\widetilde{\alpha}+1)D}(P)$$

from which we can deduce, using [Lee16, Lemma 4.1], that $(\widetilde{\alpha}+1)D - (f^*D + D) = \widetilde{\alpha}D - f^*D$ is nef, which is impossible.

A similar argument, using the $\mathbb{Q}$-divisor $f^*D - \lambda D$, shows that one cannot have $\widehat{h}_{A,D}(f(P)) \geq \widetilde{\alpha} \cdot \widehat{h}_{A,D}(P)$ for some $\widetilde{\alpha} > \alpha_D^-(f)$ and every $P \in A(\overline{\mathbb{Q}})$. □

Remark 2.38. Assume that A is simple. If the endomorphism algebra $\mathrm{End}^0(A)$ is a totally real number field, a totally definite quaternion algebra or a CM field, then the Albert classification [Mum08, Theorem 2 (p.186)] implies that there is a unique positive involution on $\mathrm{End}^0(A)$. Thus, the Rosati involution associated with any line bundle must be equal to this unique positive involution. Hence, this proves that in those cases the constants $\alpha_D^-(f), \alpha_D^+(f)$ do not depend on D.

Since all the eigenvalues of $\rho_a(f^\dagger f)$ are real and non-negative,

$$\mathrm{tr}(\rho_a(f^\dagger f)) = \alpha_1 + \ldots + \alpha_g \geq \max\{\alpha_1, \ldots, \alpha_g\} = \alpha_D^+(f)$$

so we also have the following consequence.

Corollary 2.39. *Fix an abelian variety A defined over $\overline{\mathbb{Q}}$ with an ample symmetric divisor D. Then, for every endomorphism $f : A \to A$, we have that*

$$\widehat{h}_{A,D}(f(P)) \leq \mathrm{tr}(\rho_a(f^\dagger f)) \cdot \widehat{h}_{A,D}(P)$$

for every $P \in A(\overline{\mathbb{Q}})$.

2.6.3 Height bounds for isogenies between abelian varieties

We can now generalize Theorem 2.33 to isogenies between different abelian varieties. As before, it is straightforward to see that the ratio $\widehat{h}_{B,D_2}(\phi(P))/\widehat{h}_{A,D_1}(P)$ must be bounded for non-torsion points $P \in A(\overline{\mathbb{Q}})$ (see for example [Mas84, Lemma 16] for the upper bound). The following result will make this bound explicit.

Theorem 2.40. *Let A, B be two abelian varieties defined over $\overline{\mathbb{Q}}$ and consider two ample symmetric divisors D_1, D_2 on A and B, respectively. Let also $\phi : A \to B$ be an isogeny. Then there are explicit constants $0 < \gamma_1 \leq \gamma_2$ such that*

$$\gamma_1 \cdot \widehat{h}_{A,D_1}(P) \leq \widehat{h}_{B,D_2}(\phi(P)) \leq \gamma_2 \cdot \widehat{h}_{A,D_1}(P)$$

for every $P \in A(\overline{\mathbb{Q}})$.

Proof. If π_1, π_2 are the projections of $A \times B$ onto A and B respectively, we consider the divisor $D = \pi_1^* D_1 + \pi_2^* D_2$ on $A \times B$, which is again ample and symmetric.

By the functorial properties of the canonical height, we have that

$$\widehat{h}_{A\times B,D}(P,Q) = \widehat{h}_{A\times B,\pi_1^* D_1}(P,Q) + \widehat{h}_{A\times B,\pi_2^* D_2}(P,Q) = \widehat{h}_{A,D_1}(P) + \widehat{h}_{B,D_2}(Q)$$

for every $(P,Q) \in (A \times B)(\overline{\mathbb{Q}})$.

Let also f be the endomorphism of $A \times B$ defined as $f(P,Q) = (O_A, \phi(P))$. We can then apply Theorem 2.33 to get that

$$\begin{aligned}\widehat{h}_{B,D_2}(\phi(P)) &= \widehat{h}_{A\times B,D}(f(P,Q)) \\ &\leq \alpha_D^+(f) \cdot \widehat{h}_{A\times B,D}(P,Q) = \alpha_D^+(f) \cdot \left(\widehat{h}_{A,D_1}(P) + \widehat{h}_{B,D_2}(Q)\right).\end{aligned}$$

Since this inequality holds for arbitrary $P \in A(\overline{\mathbb{Q}})$ and $Q \in B(\overline{\mathbb{Q}})$, we can choose $Q = O_B$ and thus we have

$$\widehat{h}_{B,D_2}(\phi(P)) \leq \alpha_D^+(f) \cdot \widehat{h}_{A,D_1}(P)$$

so that we can choose $\gamma_2 = \alpha_D^+(f)$. Note that this constant is the best possible, in light of Theorem 2.33.

For the lower bound, let $e(\phi)$ be the exponent of the finite group $\ker \phi$, i.e. $e(\phi)$ is the smallest positive integer n such that $[n]P = O_A$ for every $P \in \ker \phi$. Then, by [BL04, Proposition 1.2.6], there exists a unique isogeny $\psi : B \to A$ such that $\psi \circ \phi = [e(\phi)]_A$ and $\phi \circ \psi = [e(\phi)]_B$. We then apply Theorem 2.33 to the endomorphism g of $A \times B$ such that $g(P,Q) = (\psi(Q), O_B)$ in order to get

$$\begin{aligned}\widehat{h}_{A,D_1}(\psi(Q)) &= \widehat{h}_{A\times B,D}(g(P,Q)) \\ &\leq \alpha_D^+(g) \cdot \widehat{h}_{A\times B,D}(P,Q) = \alpha_D^+(g) \cdot \left(\widehat{h}_{A,D_1}(P) + \widehat{h}_{B,D_2}(Q)\right).\end{aligned}$$

As before, this implies that

$$\widehat{h}_{A,D_1}(\psi(Q)) \leq \alpha_D^+(g) \cdot \widehat{h}_{B,D_2}(Q)$$

for every $Q \in B(\overline{\mathbb{Q}})$. Then, for each $P \in A(\overline{\mathbb{Q}})$ we can choose $Q = \phi(P)$. Thus, the inequality above becomes

$$e(\phi)^2 \cdot \widehat{h}_{A,D_1}(P) = \widehat{h}_{A,D_1}((\psi \circ \phi)(P)) \leq \alpha_D^+(g) \cdot \widehat{h}_{B,D_2}(\phi(P))$$

since D_1 is symmetric. Therefore, we can take $\gamma_1 = \frac{e(\phi)^2}{\alpha_D^+(g)}$. □

Applying this theorem with $B = A$ and $\phi = [1]$ the identity gives the following comparison of canonical heights defined by different divisors (see also [HS13, Exercise B.3] for a slightly more general but ineffective statement).

Corollary 2.41. *Let A be an abelian variety defined over $\overline{\mathbb{Q}}$ and consider two ample symmetric divisors D_1, D_2 on A . Then there are explicit constants $0 < \gamma_1 \leq \gamma_2$ such that*

$$\gamma_1 \cdot \widehat{h}_{A,D_1}(P) \leq \widehat{h}_{A,D_2}(P) \leq \gamma_2 \cdot \widehat{h}_{A,D_1}(P)$$

for every $P \in A(\overline{\mathbb{Q}})$.

Lastly, we consider the special case $g = 1$. Given an elliptic curve E, a symmetric ample divisor D and an endomorphism $f \in \mathrm{End}(E)$, we clearly have $\alpha_D^-(f) = \alpha_D^+(f) = \deg f$, since $f^\dagger = \widehat{f}$. Thus, Theorem 2.33 reduces to the well known identity $\widehat{h}_{E,D}(f(P)) = \deg f \cdot \widehat{h}_{E,D}(P)$ (see for example Section 3.6 of [Ser97]).

However, for elliptic curves we may strengthen Theorem 2.40, getting again an identity instead of an inequality. We prove this using a different method from the one used before.

Proposition 2.42. *Let E_1, E_2 be two elliptic curves defined over $\overline{\mathbb{Q}}$, D_1, D_2 be two ample symmetric divisors on E_1, E_2, respectively, and $f : E_1 \to E_2$ be an isogeny. If we denote by $\widehat{h}_{E_1,D_1}$ and $\widehat{h}_{E_2,D_2}$ the canonical heights defined by the divisors D_1 and D_2, we then have*

$$\widehat{h}_{E_2,D_2}(f(P)) = \frac{\deg D_2}{\deg D_1} \cdot \deg f \cdot \widehat{h}_{E_1,D_1}(P)$$

for every $P \in E_1(\overline{\mathbb{Q}})$.

Proof. Let $a = \deg D_2 \cdot \deg f$ and $b = \deg D_1$. Then, we have

$$\deg(aD_1 - bf^*D_2) = a \cdot \deg D_1 - b \cdot \deg f \cdot \deg D_2 = 0.$$

So the divisor $aD_1 - bf^*D_2$ on E_1 is nef. As noted in Remark 2.30, the canonical height associated to a nef symmetric divisor is nonnegative, therefore

$$\begin{aligned} a \cdot \widehat{h}_{E_1,D_1}(P) - b \cdot \widehat{h}_{E_2,D_2}(f(P)) &= \widehat{h}_{E_1,aD_1}(P) - \widehat{h}_{E_1,bf^*D_2}(P) \\ &= \widehat{h}_{E_1,aD_1 - b\cdot f^*D_2}(P) \geq 0 \end{aligned}$$

implying that

$$\widehat{h}_{E_2,D_2}(f(P)) \leq \frac{\deg D_2}{\deg D_1} \cdot \deg f \cdot \widehat{h}_{E_1,D_1}(P)$$

since ample divisors on curves have positive degree [Har77, Corollary 3.3].

Similarly, $\deg(bf^*D_2 - aD_1) = 0$, so that the same argument gives

$$\widehat{h}_{E_2,D_2}(f(P)) \geq \frac{\deg D_2}{\deg D_1} \cdot \deg f \cdot \widehat{h}_{E_1,D_1}(P)$$

concluding the proof. □

Remark 2.43. Since any ample symmetric divisor on an elliptic curve is linearly equivalent to $nO + T$, where O is the identity element, n is a positive integer and T is a 2-torsion point, one can also prove Proposition 2.42 more directly, by explicitly computing the pull-back $f^*(nO+T)$ (see for example [Fer24, Proposition 2.3] for the special case $D_1 = 3(O_1)$ and $D_2 = 3(O_2)$).

Chapter 3

Unlikely intersections in families of elliptic curves

3.1 Introduction

Let m and n be positive integers. Denote by E_λ the elliptic curve with Legendre equation

$$Y^2Z = X(X-Z)(X-\lambda Z)$$

and consider this as a family of elliptic curves $E_\lambda \to Y(2) = \mathbb{A}^1 \setminus \{0,1\}$. With a slight abuse of notation, we will denote by E_λ^m the m-fold fibered power $E_\lambda \times_{Y(2)} \dots \times_{Y(2)} E_\lambda$, which defines another family $E_\lambda^m \to Y(2)$. In this chapter we will work with the product

$$E_\lambda^m \times E_\mu^n \xrightarrow{\pi} Y(2) \times Y(2).$$

Here, $E_\mu \to Y(2)$ is the Legendre family with parameter μ.

Take an irreducible curve $\mathcal{C} \subseteq E_\lambda^m \times E_\mu^n$, defined over a number field k, not contained in a fixed fiber. Then, for each point $\mathbf{c} \in \mathcal{C}$, let $\pi(\mathbf{c}) = (\lambda(\mathbf{c}), \mu(\mathbf{c})) \in Y(2) \times Y(2)$, where λ and μ are the coordinate functions on $Y(2)^2$. Also, $\mathbf{c} in \mathcal{C}$ defines m points $P_1(\mathbf{c}), \dots, P_m(\mathbf{c})$ on the elliptic curve $E_{\lambda(\mathbf{c})}$ and n points $Q_1(\mathbf{c}), \dots, Q_n(\mathbf{c})$ on the elliptic curve $E_{\mu(\mathbf{c})}$. Let R_1 and R_2 denote the generic endomorphism rings of E_λ and E_μ when restricted to $\mathcal{C}$, respectively. In general, these are equal to $\mathbb{Z}$, except in the case where one of the elliptic curves is constant on $\mathcal{C}$ and has complex multiplication. For example, if $\lambda = \lambda_0$ is constant on $\mathcal{C}$ and E_{λ_0} has complex multiplication, then R_1 is strictly larger than $\mathbb{Z}$.

We will assume that E_λ and E_μ are not generically isogenous on $\mathcal{C}$ and that the P_i's are linearly independent over R_1 and similarly for the Q_i's. This is of course equivalent to saying that there are no generic non-trivial linear relations between the P_i's and the Q_i's. Another way of rephrasing this is to say that $\mathcal{C}$ is not contained in a proper subgroup scheme of $E_\lambda^m \times E_\mu^n \to Y(2) \times Y(2)$, again assuming that E_λ and E_μ are not generically isogenous on $\mathcal{C}$.

We define the map

$$J : Y(2) \longrightarrow Y(1) = \mathbb{A}^1$$
$$\lambda \longmapsto 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

which sends λ to the j-invariant of E_λ. With a slight abuse of notation, we will also denote by J the map $Y(2)^2 \to Y(1)^2$ obtained by applying J component-wise.

Definition 3.1. Let $C \subseteq \mathbb{A}^2$ be an irreducible curve and let X, Y be the coordinate functions on $\mathbb{A}^2$. We say that C is *asymmetric* (see [Hab10]) if $\deg(X|_C) \neq \deg(Y|_C)$. Here, by convention, we set the degree of a constant map to be 0.

If $\mathcal{C} \subseteq E_\lambda^m \times E_\mu^n \xrightarrow{\pi} Y(2) \times Y(2)$ is an irreducible curve, we say that $\mathcal{C}$ is *asymmetric* if the curve $\widetilde{\mathcal{C}} = (J \circ \pi)(\mathcal{C}) \subseteq \mathbb{A}^2$ is asymmetric.

We are now ready to state the main result of this chapter.

Theorem 3.2. *Let $\mathcal{C} \subseteq E_\lambda^m \times E_\mu^n$ be an irreducible asymmetric curve defined over $\overline{\mathbb{Q}}$ not contained in a fixed fiber, and define P_i, Q_j as above. Suppose moreover that E_λ and E_μ are not generically isogenous on $\mathcal{C}$ and that there are no generic non-trivial relations among $P_1, \dots, P_m$ on E_λ and among $Q_1, \dots, Q_n$ on E_μ with coefficients in R_1 and R_2, respectively. Then, there are at most finitely many $\mathbf{c} \in \mathcal{C}(\mathbb{C})$ such that there exist an isogeny $\phi : E_{\mu(\mathbf{c})} \to E_{\lambda(\mathbf{c})}$ and $(a_1, \dots, a_{m+n}) \in \mathrm{End}(E_{\lambda(\mathbf{c})})^{m+n} \setminus \{\mathbf{0}\}$ with*

$$a_1 P_1(\mathbf{c}) + \ldots + a_m P_m(\mathbf{c}) + a_{m+1}\phi(Q_1(\mathbf{c})) + \ldots + a_{m+n}\phi(Q_n(\mathbf{c})) = O.$$

Notice that this theorem is a special case of the Zilber–Pink Conjecture. In combination with the results of [BC16], [BC17], [Bar19], and [HP16], and including the case in which one factor has complex multiplication and a linear relation holds among the points on the other factor (which will be addressed in future work), it yields a proof of the conjecture for asymmetric curves in $E_\lambda^m \times E_\mu^n$ defined over $\overline{\mathbb{Q}}$. For an account on the Zilber–Pink conjecture and other problems of Unlikely Intersections, see [Zan12] and [Pil22].

Remark 3.3. Notice that if E_λ and E_μ are generically isogenous, then $\widetilde{\mathcal{C}} = Y_0(N)$ (for some $N \geq 1$) which is not asymmetric, since the modular polynomials are symmetric (see subsection 3.2.1), and therefore have equal degree in both variables. Thus, in principle, the assumption that E_λ and E_μ are not generically isogenous could be removed from the theorem. However, in view of the Zilber–Pink conjecture, we expect that the theorem should remain valid even without the asymmetry assumption. For these reasons, in anticipation of a possible generalization of this result beyond the asymmetric setting, we prefer to leave the statement as it is.

Depending on $\pi(\mathcal{C}) \subseteq Y(2)^2$, we can distinguish three cases:

(i) the coordinate functions λ, μ on $\mathcal{C}$ are both non-constant;

(ii) (exactly) one between λ and μ is constant and the associated elliptic curve is not CM;

(iii) (exactly) one between λ and μ is constant and the associated elliptic curve is CM.

For each $\mathbf{c} \in \mathcal{C}(\mathbb{C})$, let $\rho(\mathbf{c}) \in \mathbb{C}$ be such that $\mathrm{End}(E_{\lambda(\mathbf{c})}) \cong \mathbb{Z}[\rho(\mathbf{c})]$.

In case (i), by a theorem by André [And98], there are only finitely many $\mathbf{c} \in \mathcal{C}(\overline{\mathbb{Q}})$ such that $E_{\lambda(\mathbf{c})}$ and $E_{\mu(\mathbf{c})}$ both have complex multiplication. So, recalling that isogenous elliptic curves have the same endomorphism algebra, we can discard those finitely many points and assume that $\rho = 0$ and $\mathbf{a} \in \mathbb{Z}^{m+n} \setminus \{\mathbf{0}\}$.

Similarly, in case (ii), we can assume without loss of generality that $\lambda = \lambda_0$ is constant with E_{λ_0} not CM. Therefore, there are no points $\mathbf{c} \in \mathcal{C}(\overline{\mathbb{Q}})$ such that $E_{\lambda(\mathbf{c})}$ and $E_{\mu(\mathbf{c})}$ both have complex multiplication, so we can take $\rho = 0$ and $\mathbf{a} \in \mathbb{Z}^{m+n} \setminus \{\mathbf{0}\}$ in this case as well.

In case (iii), we can assume again that $\lambda = \lambda_0$ is constant. However, in this case there are infinitely many points $\mathbf{c} \in \mathcal{C}(\overline{\mathbb{Q}})$ such that $E_{\lambda(\mathbf{c})} = E_{\lambda_0}$ and $E_{\mu(\mathbf{c})}$ are both CM, so we cannot simplify our hypothesis as before. On the other hand, since λ is constant, we can choose ρ to be a generator of $\mathrm{End}(E_{\lambda_0}) \cong \mathbb{Z}[\rho]$.

Our proof of Theorem 3.2 follows the general strategy first introduced by Pila and Zannier in [PZ08] and later used, among the others, by Masser and Zannier [MZ10, MZ12] and by Barroero and Capuano [BC16, Bar19, BC17, BC20]. In what follows, we sketch the argument only in case (i); the proofs of cases (ii) and (iii) rely on the same strategy, although their implementation requires additional technical refinements.

Since the elliptic curves E_λ and E_μ are analytically isomorphic to the complex tori $\mathbb{C}/\Lambda_{\tau_1}$ and $\mathbb{C}/\Lambda_{\tau_2}$, where $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$, with τ in the complex upper half-plane $\mathbb{H}$, we can consider the elliptic logarithms $z_1 \dots, z_m$ of $P_1, \dots, P_m$ and $w_1, \dots, w_n$ of $Q_1, \dots, Q_n$ and define a uniformization map $(\tau_1, z_1 \dots, z_m, \tau_2, w_1, \dots, w_n) \mapsto (\lambda, P_1, \dots, P_m, \mu, Q_1, \dots, Q_n)$. By a work of Peterzil and Starchenko, after restricting to a suitable fundamental domain, this map is definable in the o-minimal structure $\mathbb{R}_{\text{an, exp}}$, so the preimage of $\mathcal{C}$ is a definable surface S.

Let $\mathcal{C}'$ be the subset of $\mathcal{C}$ we want to prove to be finite. Then, the points $\mathbf{c}_0 \in \mathcal{C}'$ correspond to points on S lying on subvarieties defined by equations with integer coefficients. We then use a result by Habegger and Pila, which implies that there are $\ll T^\varepsilon$ points of S lying on the subvarieties with coefficients bounded in absolute value by T, provided that the z_i and the w_j are algebraically independent over $\mathbb{C}(\tau_1, \tau_2)$.

We then use a result by Habegger [Hab10] for asymmetric curves[1], giving height bounds for $\lambda(\mathbf{c}_0)$, $\mu(\mathbf{c}_0)$, the $P_i(\mathbf{c}_0)$ and the $Q_j(\mathbf{c}_0)$. By a result of Masser [Mas88], these bounds imply that the coefficients $a_1, \ldots, a_{m+n}$ of the linear relation between the $m+n$ points

$$P_1(\mathbf{c}_0), \ldots, P_m(\mathbf{c}_0), \phi(Q_1(\mathbf{c}_0)), \ldots, \phi(Q_n(\mathbf{c}_0))$$

can be taken to be bounded by a constant times a positive power of $D_0 = [k(\lambda(\mathbf{c}_0), \mu(\mathbf{c}_0)) : k]$. Moreover, all Galois conjugates of $\mathbf{c}_0$ are still in $\mathcal{C}'$, so that we have at least D_0 points on S lying on the subvarieties with coefficients bounded in absolute value by some positive power of D_0. Combining this with the previous bound, we get that D_0 is bounded and therefore the claim of the theorem, by Northcott's theorem.

Remark 3.4. Let $C \subseteq E_\lambda^m \times E_\mu^n$ be an irreducible curve not contained in a fixed fiber, not necessarily asymmetric, and define P_i, Q_j and $\tau_1, z_1, \ldots, z_m, \tau_2, w_1, \ldots, w_n$ as above (see also Section 3.2.2 for more details). Assume also that E_λ and E_μ are not generically isogenous on C and that there are no generic non-trivial relations among $P_1, \ldots, P_m$ on E_λ and among $Q_1, \ldots, Q_n$ on E_μ with coefficients in R_1 and R_2, respectively.

Then, in case (i) let $\ell = 0$, while in case (ii) and (iii) let $\ell \geq 0$ be the greatest integer such that there are $\widetilde{a}_{i,j} \in \operatorname{End}(E_{\lambda_0})$ and $\widetilde{P}_j \in E_{\lambda_0}(\overline{\mathbb{Q}})$, $i = 1, \ldots, m$ and $j = 1, \ldots, \ell$, such that the vectors $\widetilde{\mathbf{a}}_j := (\widetilde{a}_{1,j}, \ldots, \widetilde{a}_{m,j})$, for $j = 1, \ldots, \ell$, are $\operatorname{End}(E_{\lambda_0})$-linearly independent and

$$\widetilde{a}_{1,j} P_1(\mathbf{c}) + \ldots + \widetilde{a}_{m,j} P_m(\mathbf{c}) = \widetilde{P}_j$$

for every $\mathbf{c} \in C$. In particular, the assumption that there is no generic non-trivial relation among the P_i and the assumption on the vectors $\widetilde{\mathbf{a}}_1, \ldots, \widetilde{\mathbf{a}}_\ell$ implies that $\widetilde{P}_1, \ldots, \widetilde{P}_\ell$ are $\operatorname{End}(E_{\lambda_0})$-linearly independent. Up to reordering the P_i, we can then assume that the matrix $(\widetilde{a}_{i,j})_{i,j=1,\ldots,\ell}$ has maximal rank. Hence, we can consider the isogeny

$$\begin{aligned} \Phi : E_{\lambda_0}^m \times E_\mu^n &\longrightarrow E_{\lambda_0}^m \times E_\mu^n \\ (P_1, \ldots, P_m, Q_1, \ldots, Q_n) &\longmapsto \left(\sum_{i=1}^m \widetilde{a}_{i,1} P_i, \ldots, \sum_{i=1}^m \widetilde{a}_{i,\ell} P_i, P_{\ell+1}, \ldots, P_m, Q_1, \ldots, Q_n \right) \end{aligned}$$

which sends $(\mathbf{P}, \mathbf{Q}) \in C$ to $(\widetilde{P}_1, \ldots, \widetilde{P}_\ell, P_{\ell+1}, \ldots, P_m, \mathbf{Q}) \in \Phi(C)$. Note that the maximality of ℓ also implies that $P_{\ell+1}, \ldots, P_m$ are generically $\operatorname{End}(E_{\lambda_0})$-linearly independent modulo constants, i.e. there is no relation of the form

$$a_{\ell+1} P_{\ell+1}(\mathbf{c}) + \ldots + a_m P_m(\mathbf{c}) = \widetilde{P}$$

with $a_{\ell+1}, \ldots, a_m \in \operatorname{End}(E_{\lambda_0})$ not all zero and $\widetilde{P} \in E_{\lambda_0}(\overline{\mathbb{Q}})$, that holds for every $\mathbf{c} \in C$.

[1]This is the only step of the proof where we use the assumption on the asymmetry of $\mathcal{C}$, see also remark 3.15.

Notice that if C satisfies the hypotheses of Theorem 3.2, then $\Phi(C)$ satisfies them as well, and vice versa. Moreover, as the restriction of Φ to any fiber is again an isogeny, images and preimages under Φ of algebraic subgroups of a fiber are again algebraic subgroups. This implies that Theorem 3.2 holds for C if and only if it holds for $\Phi(C)$.

Therefore, up to applying the isogeny Φ, we will always assume that on the (asymmetric) curve $\mathcal{C}$ that we are considering in Theorem 3.2, $P_1, \ldots, P_\ell$ are constant and $\mathrm{End}(E_{\lambda_0})$-linearly independent, as above.

Remark 3.5. I am grateful to Gabriel Dill for pointing out that cases (ii) and (iii) can also be deduced from Theorem 1.2 in [Dil21]. Continuing with the notation introduced in the previous remark and using the notation from [Dil21], take $A_0 = E_{\lambda_0}^{m-\ell+n}$, $\mathcal{A} = E_{\lambda_0}^{m-\ell} \times E_\mu^n$ and $\Gamma = (\Gamma_0)^{m-\ell+n}$, where Γ_0 is the divisible hull of the subgroup of $E_{\lambda_0}(\overline{\mathbb{Q}})$ generated by $\mathrm{End}(E_{\lambda_0}) \cdot \widetilde{P}_1, \ldots, \mathrm{End}(E_{\lambda_0}) \cdot \widetilde{P}_\ell$. Then, if C is the projection of $\Phi(\mathcal{C})$ onto $\mathcal{A}$, $\mathcal{A}_\Gamma^{[1]} \cap C$ consists exactly of the points described in Theorem 3.2 and, by [Dil21, Theorem 1.2], we get that either this intersection is finite or that the generic fiber $C_\xi \subset C$ is contained in the translate of a proper abelian subvariety of $\mathcal{A}_\xi$ by a point in

$$(\mathcal{A}_\xi)_{\mathrm{tors}} + \mathrm{Tr}(\mathcal{A}_\xi) = E_{\lambda_0}^{m-\ell}(\overline{\mathbb{Q}}) \times (E_\mu^n)_{\mathrm{tors}}.$$

However, the latter means that either $Q_1, \ldots, Q_n$ are generically linearly dependent or that there is a non trivial linear relation modulo constants involving $P_{\ell+1}, \ldots, P_m$, contradicting our assumptions.

We use Vinogradov's $\ll$ notation: for real-valued functions $f(T)$ and $g(T)$, we write $f(T) \ll g(T)$ if there exists a constant $\gamma > 0$ such that $f(T) \leq \gamma g(T)$ for all sufficiently large T. When not explicitly stated, the implied constant is either absolute or depends only on $\mathcal{C}$ and other fixed data. We use subscripts to indicate any additional dependence of the implied constant.

3.2 Preliminaries

3.2.1 Isogenies and modular curves

Let $E_1 \cong \mathbb{C}/\Lambda_1$ and $E_2 \cong \mathbb{C}/\Lambda_2$ be two elliptic curves defined over $\mathbb{C}$. Up to homothety, any lattice in $\mathbb{C}$ is of the form $\mathbb{Z} + \mathbb{Z}\tau$ for some τ in the upper half-plane $\mathbb{H}$, and any two such lattices define isomorphic complex tori if and only if the corresponding parameters τ lie in the same orbit under the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ by fractional linear transformations. Therefore, we may choose $\tau_1, \tau_2 \in \mathbb{H}$ such that $\Lambda_1 = \mathbb{Z} + \mathbb{Z}\tau_1$ and $\Lambda_2 = \mathbb{Z} + \mathbb{Z}\tau_2$, with τ_1 and τ_2 lying in the standard fundamental domain $\mathfrak{F} \subseteq \mathbb{H}$ for the action of $\mathrm{SL}_2(\mathbb{Z})$. This domain is given by

$$\mathfrak{F} = \left\{ \tau \in \mathbb{H} : |\tau| \geq 1, -\frac{1}{2} \leq \mathrm{Re}(\tau) < \frac{1}{2} \right\} \setminus \left\{ \tau \in \mathbb{H} : |\tau| = 1, 0 < \mathrm{Re}(\tau) < \frac{1}{2} \right\}. \tag{3.1}$$

In particular, the choice of $\tau_1, \tau_2 \in \mathfrak{F}$ ensures that the associated elliptic curves are uniquely determined up to isomorphism.

Recall that for each isogeny $\phi : E_1 \to E_2$ there exists a unique non-zero complex number α such that $\alpha\Lambda_1 \subseteq \Lambda_2$ and ϕ corresponds to the multiplication-by-α map $\mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$.

Therefore, if E_1 and E_2 are isogenous, then there exists $\alpha \in \mathbb{C} \setminus \{0\}$ and integers A, B, C, D not all zero (not necessarily coprime) such that

$$\alpha \cdot \tau_1 = A\tau_2 + B$$
$$\alpha \cdot 1 = C\tau_2 + D$$

thus

$$\tau_1 = \frac{A\tau_2 + B}{C\tau_2 + D}.$$

Moreover, the converse is also true. If $\tau_1, \tau_2 \in \mathbb{H}$ and $\tau_1 = \frac{A\tau_2+B}{C\tau_2+D}$ for integers A, B, C, D, then there exists an isogeny $\phi : E_1 \to E_2$ corresponding to $\alpha = C\tau_2 + D$.

More generally, we have an action of the group $\mathrm{GL}_2^+(\mathbb{Q})$ (here $+$ means that the matrices have positive determinant) on the upper half-plane $\mathbb{H}$ which is given by

$$M\tau = \frac{a\tau + b}{c\tau + d}$$

for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$. If $M \in \mathrm{Mat}(\mathbb{Z}, 2)$, we say that M is *primitive* if $\gcd(a, b, c, d) = 1$.

We say that an isogeny ϕ is *cyclic* if $\ker \phi$ is a (finite) cyclic group. Then it is known (see [DS05, Section 1.3]) that any isogeny can be written as the composition of a cyclic isogeny and a multiplication-by-n isogeny, for some integer n. In particular, cyclic isogenies $E_1 \to E_2$ correspond to relations $\tau_1 = M\tau_2$ with M primitive. In this case, the degree of the isogeny is equal to $\det M$.

Recall also that the modular polynomials $\Phi_N(X, Y) \in \mathbb{Z}[X, Y]$ are the irreducible symmetric polynomials parametrizing pairs of isomorphism classes of elliptic curves with a cyclic isogeny of degree N between them [Lan87, Chapter 5]. In other words, $\Phi_N(j_1, j_2) = 0$ if and only if there exists a cyclic isogeny of degree N between the elliptic curves with j-invariants j_1 and j_2. We then define the classical modular curve $Y_0(N) \subset \mathbb{A}^2$ as the plane curve defined by the equation $\Phi_N(X, Y) = 0$.

Finally, the following result provides an effective bound for the size of the integers A, B, C, D when the degree of the isogeny is fixed. It is a consequence of Theorem 1.1 of

[Orr18] (see Section 1.A therein for details) and constitutes an improvement of Lemma 5.2 of [HP12].

Lemma 3.6. *There exists an absolute constant $c > 0$ with the following property: if $E_1 = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau_1), E_2 = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau_2)$ are elliptic curves with $\tau_1, \tau_2 \in \mathfrak{F}$, and there exists a cyclic isogeny $\phi : E_1 \to E_2$ of degree N, then there are integers A, B, C, D such that*

$$AD - BC = N \qquad \tau_1 = \frac{A\tau_2 + B}{C\tau_2 + D} \qquad |A|, |B|, |C|, |D| \leq cN.$$

3.2.2 Uniformization

Let $\mathcal{A}$ be the quasi-projective variety in $Y(2) \times (\mathbb{P}^2)^m \times Y(2) \times (\mathbb{P}^2)^n$ with coordinates

$$(\lambda, [X_1 : Y_1 : Z_1], \ldots, [X_m : Y_m : Z_m], \mu, [U_1 : V_1 : W_1], \ldots, [U_n : V_n : W_n])$$

and defined by the $n + m$ equations

$$\begin{aligned} Y_i^2 Z_i &= X_i(X_i - Z_i)(X_i - \lambda Z_i) & i &= 1, \ldots, m \\ V_j^2 W_j &= U_j(U_j - W_j)(U_j - \mu W_j) & j &= 1, \ldots, n. \end{aligned}$$

We set $P_i = [X_i : Y_i : Z_i]$ and $Q_j = [U_j : V_j : W_j]$ and we have an irreducible curve $C \subseteq \mathcal{A}$ defined over a number field k such that the projection of $\mathcal{A}$ to $Y(2) \times Y(2)$ restricts to rational functions λ and μ on $\mathcal{C}$ not both constant.

The aim of this section is to define a uniformization map for $\mathcal{A}$, following closely the exposition in [Pil09, pp. 2489–2491].

As said before, any elliptic curve over $\mathbb{C}$ is analytically isomorphic to a complex torus $\mathbb{C}/\Lambda_\tau$, where τ has positive imaginary part and Λ_τ is the lattice generated by 1 and τ, with fundamental domain

$$\mathcal{L}_\tau = \{z \in \mathbb{C} : z = x + \tau y, x, y \in [0, 1)\}.$$

The classical Weierstrass $\wp$-function $\wp(z, \Lambda_\tau) = \wp(z, \tau)$ associated to the lattice Λ_τ, is Λ_τ-periodic and satisfies the following differential equation

$$(\wp(z, \tau)')^2 = 4\wp(z, \tau)^3 - g_2(\tau)\wp(z, \tau) - g_3(\tau)$$

where $\wp(z, \tau)' = \frac{d}{dz}\wp(z, \tau)$ and $g_2(\tau), g_3(\tau)$ are defined in [Sil09, Remark 3.5.1]. Then, the zeros of the polynomial $4X^3 - g_2(\tau)X - g_3(\tau)$ are exactly the values of $\wp$ at the half-periods:

$$e_1(\tau) = \wp\left(\frac{1}{2}, \tau\right) \quad e_2(\tau) = \wp\left(\frac{1 + \tau}{2}, \tau\right) \quad e_3(\tau) = \wp\left(\frac{\tau}{2}, \tau\right).$$

Note that the $e_i(\tau)$ are pairwise distinct (see [Sil09, Proposition VI.3.6] and [For51, Sec. 63]) and that the function $e_3 - e_1$ has a regular square root on all of $\mathbb{H}$. Therefore, we can define

$$\xi(z,\tau) = \frac{\wp(z,\tau) - e_1(\tau)}{e_3(\tau) - e_1(\tau)} \quad \text{and} \quad \eta(z,\tau) = \frac{\wp(z,\tau)'}{2(e_3(\tau) - e_1(\tau))^{\frac{3}{2}}}$$

so that we have the following relation

$$\eta(z,\tau)^2 = \xi(z,\tau)(\xi(z,\tau) - 1)(\xi(z,\tau) - L(\tau))$$

where

$$L(\tau) = \frac{e_2(\tau) - e_1(\tau)}{e_3(\tau) - e_1(\tau)}. \tag{3.2}$$

This gives a parametrization of the Legendre family via the map $(z,\tau) \mapsto (L(\tau), P(z,\tau))$, where

$$P(z,\tau) = \begin{cases} [\xi(z,\tau) : \eta(z,\tau) : 1] & \text{if } z \notin \Lambda_\tau \\ [0:1:0] & \text{otherwise} \end{cases}$$

Finally, define the map $\varphi : \mathbb{H}\times\mathbb{C}^m\times\mathbb{H}\times\mathbb{C}^n \to \mathcal{A}(\mathbb{C})$ sending $(\tau_1, z_1, \ldots, z_m, \tau_2, w_1, \ldots, w_n)$ to $(L(\tau_1), P(z_1,\tau_1), \ldots, P(z_m,\tau_1), L(\tau_2), P(w_1,\tau_2), \ldots, P(w_n,\tau_2))$. Since this map is not injective, we would like to find a subset of the domain over which it is possible to define a univalued inverse function of φ.

By [For51, Sec. 70], there exists a finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ such that $L(\gamma\tau) = L(\tau)$ for every $\gamma \in \Gamma$. Moreover, as a fundamental domain for the action of Γ on $\mathbb{H}$ one can take the union of six suitably chosen fundamental domains for the action of $\mathrm{SL}_2(\mathbb{Z})$ (see [For51, Fig. 48 and 49]). We will call this set $\mathcal{B}$ and define

$$\mathcal{F}_\mathcal{B} = \{(\tau_1, z_1, \ldots, z_m, \tau_2, w_1, \ldots, w_n) : \tau_1, \tau_2 \in \mathcal{B}, z_1, \ldots, z_m \in \mathcal{L}_{\tau_1}, w_1, \ldots, w_n \in \mathcal{L}_{\tau_2}\}.$$

Then, φ has a univalued inverse $\mathcal{A}(\mathbb{C}) \to \mathcal{F}_\mathcal{B}$ and we set

$$\mathcal{Z} = \varphi^{-1}(\mathcal{C}(\mathbb{C})) \cap \mathcal{F}_\mathcal{B}. \tag{3.3}$$

Following Remark 3.4, we assume that $P_1 = \widetilde{P}_1, \ldots, P_\ell = \widetilde{P}_\ell$ are constant on $\mathcal{C}$, so that $\mathcal{Z}$ consists of points of the form $(\tau_1, \widetilde{z}_1, \ldots, \widetilde{z}_\ell, z_{\ell+1}, \ldots, z_m, \tau_2, w_1, \ldots, w_n)$, where $\widetilde{z}_i$ is the (constant) elliptic logarithm of the constant point $\widetilde{P}_i$.

Having described the uniformization of $\mathcal{A}$, we now turn to a key result of functional transcendence that will play an essential role in our arguments. Let $\widehat{\mathcal{C}}$ be the subset of the smooth points of $\mathcal{C}(\mathbb{C})$ that are not ramified points of $\pi_{|\mathcal{C}}$. In this way, the set $\mathcal{C}(\mathbb{C}) \setminus \widehat{\mathcal{C}}$ consists only of finitely many algebraic points of $\mathcal{C}$.

Fix a point $\mathbf{c}_* \in \widehat{\mathcal{C}}$ and let $D_{\mathbf{c}_*} \subseteq \widehat{\mathcal{C}}$ be a small open disc containing $\mathbf{c}_*$. Let $\mathbf{t}_* = \varphi^{-1}_{|\mathcal{F}_\mathcal{B}}(\mathbf{c}_*)$. Then, there exists an open connected neighbourhood $U_* \subseteq \mathbb{H}\times\mathbb{C}^m\times\mathbb{H}\times\mathbb{C}^n$ of $\mathbf{t}_*$

such that $\varphi(U_*) = D_{\mathbf{c}_*}$. Thus, $\tau_1, z_1, \ldots, z_m, \tau_2, w_1, \ldots, w_n$ are well-defined holomorphic functions (possibly constant) on U_* and, with a slight abuse of notation, we consider them as holomorphic functions on $D_{\mathbf{c}_*}$.

With these definitions, we have the following transcendence result.

Lemma 3.7. *The functions $z_{\ell+1}, \ldots, z_m, w_1, \ldots, w_n$ are algebraically independent over $\mathbb{C}(\tau_1, \tau_2)$ on $D_{\mathbf{c}_*}$.*

Proof. In case (i) we can apply Corollary 2.5 from [BC17] (which is based on a result by Bertrand [Ber09]).

In case (ii) and (iii), notice that our assumption on $\mathcal{C}$ imply that

$$\mathcal{C} \subseteq \left\{ \left(\widetilde{P}_1, \ldots, \widetilde{P}_\ell \right) \right\} \times E_{\lambda_0}^{m-\ell} \times E_\mu^n.$$

Let also $F = \mathbb{C}(\tau_2)$ (note that $F = \mathbb{C}(\tau_1, \tau_2)$, since τ_1 is constant in these cases) and assume by contradiction that

$$\mathrm{tr.deg}_F F\left(z_{\ell+1}, \ldots, z_m, w_1, \ldots, w_n\right) < m + n - \ell.$$

Then, if $\widetilde{z}_1, \ldots, \widetilde{z}_\ell \in \mathcal{L}_{\tau_1}$ are elliptic logarithms of the constant points $\widetilde{P}_1, \ldots, \widetilde{P}_\ell$, on $\mathcal{C}$ we must have

$$\begin{aligned} &\mathrm{tr.deg}_F F(\widetilde{z}_1, \ldots, \widetilde{z}_\ell, z_{\ell+1}, \ldots, z_m, w_1, \ldots, w_n) = \\ &\qquad \mathrm{tr.deg}_F F\left(z_{\ell+1}, \ldots, z_m, w_1, \ldots, w_n\right) < m + n - \ell. \end{aligned}$$

Applying Theorem 7.1 from [Dil21] to $\mathcal{C}$, we obtain a subvariety $\mathcal{W} \subseteq E_{\lambda_0}^m \times E_\mu^n$ containing $\mathcal{C}$. This subvariety is a translate of an abelian subscheme of $E_{\lambda_0}^m \times E_\mu^n$ by the product of a constant section of $E_{\lambda_0}^m$ (defined over $\overline{\mathbb{Q}}$) with a torsion multisection of E_μ^n. Moreover, one has $\dim(\mathcal{W}) \leq m + n - \ell$.

Since $Q_1, \ldots, Q_n$ are linearly independent by hypothesis, this implies that there are $\widetilde{a}_{i,\ell+1} \in \mathrm{End}(E_{\lambda_0})$ and $\widetilde{P}_{\ell+1} \in E_{\lambda_0}(\overline{\mathbb{Q}})$, $i = \ell+1, \ldots, m$, such that $\widetilde{a}_{\ell+1,\ell+1}, \ldots, \widetilde{a}_{m,\ell+1}$ are not all zero and

$$\widetilde{a}_{\ell+1,\ell+1} P_{\ell+1} + \ldots + \widetilde{a}_{m,\ell+1} P_m = \widetilde{P}_{\ell+1}$$

contradicting the maximality of ℓ and proving that

$$\mathrm{tr.deg}_F F\left(z_{\ell+1}, \ldots, z_m, w_1, \ldots, w_n\right) = m + n - \ell.$$

□

3.2.3 Heights

Let h denote the logarithmic absolute Weil height on $\mathbb{P}^N$, as defined in Section 2, and, if α is an algebraic number, define $h(\alpha) = h\left([1 : \alpha]\right)$. Define also the multiplicative Weil height as $H(P) = \exp(h(P))$.

For an elliptic curve E defined over $\overline{\mathbb{Q}}$ and a point $P \in E(\overline{\mathbb{Q}}) \subseteq \mathbb{P}^2(\overline{\mathbb{Q}})$, we also have the Néron-Tate height $\widehat{h}$, defined as follows (see also [Sil09, VII.9]):

$$\widehat{h}(P) = \lim_{n\to\infty} \frac{1}{4^n} h(2^n P).$$

By Example 2.24, we know that $\widehat{h}$ is the canonical height associated with the divisor $3O$, so Proposition 2.42 implies that

$$\widehat{h}_2(\phi(P)) = \deg\phi \cdot \widehat{h}_1(P) \tag{3.4}$$

for any $P \in E_1(\overline{\mathbb{Q}})$, where $\phi : E_1 \to E_2$ is any isogeny between the two elliptic curves E_1 and E_2, defined over $\overline{\mathbb{Q}}$ and $\widehat{h}_1$ and $\widehat{h}_2$ are the Néron-Tate heights on E_1 and E_2, defined as above.

Using the same notation as in the previous section, we have that if $\mathbf{c} \in \mathcal{C}(\overline{\mathbb{Q}})$, then standard properties of heights (see [HS13, Theorem B.2.5]) imply that, if λ and μ are both non-constant, we have

$$h(P_i(\mathbf{c})) \ll h(\lambda(\mathbf{c})) + 1 \quad \text{and} \quad h(Q_j(\mathbf{c})) \ll h(\mu(\mathbf{c})) + 1 \tag{3.5}$$

for every $i = \ell+1, \ldots, m$ and $j = 1, \ldots, n$. In case (ii) and (iii), if $\lambda = \lambda_0$ is constant, we have that $h(P_i(\mathbf{c})) \ll h(\mu(\mathbf{c})) + 1$, as we can use μ as uniformizing parameter on the base $\pi(\mathcal{C}) = \{\lambda_0\} \times \mathbb{A}^1$. Moreover, note that if $\mathcal{C}$ is defined over a number field k, we also have

$$[k(\mathbf{c}) : k] \ll [k(\lambda(\mathbf{c}), \mu(\mathbf{c})) : k]\,.$$

Finally, we will also need another definition of height (from [HP12, Section 7], see also Definition 4.6 for a generalization).

Definition 3.8. If α is a complex number, we define

$$H_1(\alpha) := \begin{cases} H(\alpha) = \max\{|p|, |q|\} & \text{if } \alpha = \frac{p}{q} \in \mathbb{Q}, \gcd(p,q) = 1 \\ +\infty \quad \text{otherwise} \end{cases}$$

For $(\alpha_1, \ldots, \alpha_N) \in \mathbb{C}^N$, we also define $H_1(\alpha_1, \ldots, \alpha_N) = \max\{H_1(\alpha_i)\}$.

3.2.4 Complex Multiplication

Given a $\lambda_0 \in Y(2)$ such that E_{λ_0} has complex multiplication, we know that the associated $\tau_0 \in \mathcal{B}$ is an algebraic number of degree 2, with minimal polynomial $aX^2 + bX + c \in \mathbb{Z}[X]$ and discriminant $\Delta_0 = b^2 - 4ac < 0$. In this case, we know by [Lan87, Theorem 1, p. 90], that

$$\operatorname{End}(E_{\lambda_0}) \cong \mathbb{Z}[\rho_0] =: \mathcal{O}_{\lambda_0}$$

where $\rho_0 = \frac{\Delta_0+\sqrt{\Delta_0}}{2} \in \mathbb{C}$ and the isomorphism is given by [Sil94, Proposition II.1.1]. Using this proposition and with a slight abuse of notation, we will identify endomorphisms with the corresponding complex number.

By Theorem II.4.3. of [Sil94],

$$[\mathbb{Q}(j_0) : \mathbb{Q}] = cl(\mathcal{O}_{\lambda_0})$$

where j_0 is the j-invariant of E_{λ_0} (which is algebraic by [Sil94, Proposition II.2.1]) and $cl(\mathcal{O}_{\lambda_0})$ is the class number of $\mathcal{O}_{\lambda_0}$.

Moreover, a theorem of Siegel in the form of Theorem 1.2 of [Bre01] gives us the estimate

$$|\Delta_0|^{\frac{1}{2}-\epsilon} \ll_\epsilon cl(\mathcal{O}_{\lambda_0}) \ll_\epsilon |\Delta_0|^{\frac{1}{2}+\epsilon}$$

so that, in particular, we have $|\Delta_0| \ll [\mathbb{Q}(j_0) : \mathbb{Q}]^3$. Finally, using Equation (3.4) and the fact that the endomorphism ρ_0 has degree $(\Delta_0^2 - \Delta_0)/4$, we get that

$$\widehat{h}(\rho_0 P) \ll |\Delta_0|^2\, \widehat{h}(P) \ll [\mathbb{Q}(j_0) : \mathbb{Q}]^6\, \widehat{h}(P) \ll [\mathbb{Q}(\lambda_0) : \mathbb{Q}]^6\, \widehat{h}(P) \tag{3.6}$$

for every $P \in E_{\lambda_0}\big(\overline{\mathbb{Q}}\big)$.

3.3 O-minimality and definable sets

In this section we recall the basic properties and some results about o-minimal structures. For more details see [vdD98] and [vdDM96].

Definition 3.9. A *structure* is a sequence $\mathcal{S} = (\mathcal{S}_N)$, $N \geq 1$, where each $\mathcal{S}_N$ is a collection of subsets of $\mathbb{R}^N$ such that, for each $N, M \geq 1$:

- $\mathcal{S}_N$ is a boolean algebra (under the usual set-theoretic operations);
- $\mathcal{S}_N$ contains every semi-algebraic subset of $\mathbb{R}^N$;
- if $A \in \mathcal{S}_N$ and $B \in \mathcal{S}_M$, then $A \times B \in \mathcal{S}_{N+M}$;
- if $A \in \mathcal{S}_{M+N}$, then $\pi(A) \in \mathcal{S}_M$, where $\pi : \mathbb{R}^{M+N} \to \mathbb{R}^M$ is the projection onto the first M coordinates.

If $\mathcal{S}$ is a structure and, in addition,

- $\mathcal{S}_1$ consists of all finite unions of open intervals and points

then $\mathcal{S}$ is called an *o-minimal structure*.

Given a structure $\mathcal{S}$, we say that $S \subseteq \mathbb{R}^N$ is a *definable set* if $S \in \mathcal{S}_N$.

Given $S \subseteq \mathbb{R}^N$ and a function $f : S \to \mathbb{R}^M$, we say that f is a *definable function* if its graph $\left\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^M : x \in S, y = f(x)\right\}$ is a definable set. One can easily prove that images and preimages of definable sets via definable functions are still definable.

Let $U \subseteq \mathbb{R}^{M+N}$. For $t_0 \in \mathbb{R}^M$, we set $U_{t_0} = \left\{x \in \mathbb{R}^N : (t_0, x) \in U\right\}$ and call U a *family* of subsets of $\mathbb{R}^N$, while U_{t_0} is called the *fiber* of U above t_0. If U is a definable set, then we call it a *definable family* and it is easy to prove that the fibers U_{t_0} are also definable.

Proposition 3.10 ([vdDM96], 4.4)**.** *Let $U \subseteq \mathbb{R}^M \times \mathbb{R}^N$ be a definable family in a fixed o-minimal structure $\mathcal{S}$. Then, there exists an integer n such that, for every $t_0 \in \mathbb{R}^M$, U_{t_0} has at most n connected components.*

While there are many examples of o-minimal structures (see [vdDM96]), in this chapter we will work with the structure $\mathbb{R}_{\text{an,exp}}$ (see [Pil22, Chapter 8] for details about this structure), which was proved to be o-minimal by van den Dries and Miller [vdDM94].

For a family $Z \subseteq \mathbb{R}^M \times \mathbb{R}^N = \mathbb{R}^{M+N}$ and a positive real number T define

$$Z^{\sim}(\mathbb{Q}, T) := \left\{(y, z) \in Z : y \in \mathbb{Q}^M, H_1(y) \le T\right\}$$

where $H_1(y)$ is the 1-polynomial height defined in the previous section and let π_1, π_2 be the projections of Z to the first M and last N coordinates, respectively.

Proposition 3.11 ([HP16], Corollary 7.2)**.** *Let $Z \subseteq \mathbb{R}^{M+N}$ be a definable set. For every $\varepsilon > 0$ there exists a positive constant $c = c(Z, \varepsilon)$ with the following property. If $T \ge 1$ and $|\pi_2(Z^{\sim}(\mathbb{Q}, T))| > cT^{\varepsilon}$, then there exists a continuous definable function $\delta : [0, 1] \to Z$ such that:*

1. *the restriction $\delta|_{(0,1)}$ is real analytic (since $\mathbb{R}_{\text{an, exp}}$ admits analytic cell decomposition);*
2. *the composition $\pi_1 \circ \delta : [0, 1] \to \mathbb{R}^M$ is semi-algebraic and its restriction to $(0, 1)$ is real analytic;*
3. *the composition $\pi_2 \circ \delta : [0, 1] \to \mathbb{R}^N$ is non-constant.*

Lastly, we want to prove that the set $\mathcal{Z}$ defined in (3.3) is definable in $\mathbb{R}_{\text{an,exp}}$. In the following, definability will always be considered in $\mathbb{R}_{\text{an,exp}}$, and we say that $X \subseteq \mathbb{C}^N$ is definable if the set $\{(\text{Re}(z_1), \text{Im}(z_1), \dots, \text{Re}(z_N), \text{Im}(z_N)) : (z_1, \dots, z_N) \in X\} \subseteq \mathbb{R}^{2N}$ is definable. Similarly, a function $f : X \to \mathbb{C}$ is definable if and only if $\text{Re}(f)$ and $\text{Im}(f)$ are both definable.

Let $\mathcal{D}$ be the usual fundamental domain for the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, then the restriction of $\wp(z, \tau)$ to $\{(z, \tau) : \tau \in \mathcal{D}, z \in \mathcal{L}_\tau\}$ is definable by work of Peterzil and Starchenko [PS05]. Therefore, $\wp(z, \tau)$ is definable even if restricted to $\{(z, \tau) : \tau \in \gamma\mathcal{D}, z \in \mathcal{L}_\tau\}$, for any fundamental domain $\gamma\mathcal{D}$ for $\mathrm{SL}_2(\mathbb{Z})$. Since $\mathcal{B}$ is the union of six such fundamental domains, we have that $\wp(z,\tau)$ is also definable when restricted to $\{(z,\tau):\tau\in\mathcal{B}, z\in\mathcal{L}_\tau\}$. Thus, the uniformization map φ, defined in the previous section and restricted to $\mathcal{F}_{\mathcal{B}}$, is definable. Since $\mathcal{C}(\mathbb{C})$ is semi-algebraic and $\mathcal{F}_{\mathcal{B}}$ is definable, we get that $\mathcal{Z}=\varphi^{-1}(\mathcal{C}(\mathbb{C}))\cap\mathcal{F}_{\mathcal{B}}$ is definable.

3.4 The main estimate

We continue with the notations established in the previous sections and, for every $T\geq 1$, define the set

$$\begin{aligned}\mathcal{Z}(T)=\Bigg\{&(\tau_1,\widetilde{z}_1,\ldots,\widetilde{z}_\ell,z_{\ell+1},\ldots,z_m,\tau_2,w_1,\ldots,w_n)\in\mathcal{Z}:|\tau_1|,|\tau_2|\leq T,\mathrm{Im}(\tau_1)\geq\frac{1}{T},\\&\exists A,B,C,D\in\mathbb{Z}\cap[-T,T]\ \text{with}\ AD-BC\neq 0,\tau_2=\frac{A\tau_1+B}{C\tau_1+D},\\&\exists\,(a_1,\ldots,a_{m+n},b_1,\ldots,b_{m+n})\in\mathbb{Z}^{2m+2n}\ \text{with}\ (a_{\ell+1}+b_{\ell+1}\rho,\ldots,a_{m+n}+b_{m+n}\rho)\neq\mathbf{0},\\&\max|a_i|,|b_i|\leq T\ \text{and}\\&\sum_{i=1}^{\ell}(a_i+b_i\rho)\widetilde{z}_i+\sum_{i=\ell+1}^{m}(a_i+b_i\rho)z_i+(C\tau_1+D)\sum_{j=1}^{n}(a_{m+j}+b_{m+j}\rho)w_j\in\mathbb{Z}+\mathbb{Z}\tau_1\Bigg\}\end{aligned}$$

where $\mathcal{Z}$ is the set defined in (3.3), $\widetilde{z}_1,\ldots,\widetilde{z}_\ell$ are the elliptic logarithms of the constant points $\widetilde{P}_1,\ldots,\widetilde{P}_\ell$ and ρ is either 0 in case (i) and (ii), or a fixed quadratic integer in case (iii).

The goal of this section is to prove the following result.

Proposition 3.12. *Under the hypotheses of Theorem 3.2, for all $\varepsilon>0$, we have $\#\mathcal{Z}(T)\ll_\varepsilon T^\varepsilon$, for all $T\geq 1$.*

To prove this, we will apply Proposition 3.11 to the definable set W consisting of tuples of the form

$$\begin{aligned}&(\alpha_1,\ldots,\alpha_{m+n},\beta_1,\ldots,\beta_{m+n},A,B,C,D,\xi_1,\xi_2,\\&\zeta_1,\theta_1,\widetilde{x}_1,\widetilde{y}_1,\ldots,\widetilde{x}_\ell,\widetilde{y}_\ell,x_{\ell+1},y_{\ell+1},\ldots,x_m,y_m,\zeta_2,\theta_2,u_1,v_1,\ldots,u_n,v_n)\end{aligned}$$

in $\mathbb{R}^{2m+2n+6}\times\mathbb{R}^{2m+2n+4}$, satisfying the following relations:

$$(\alpha_{\ell+1}+\beta_{\ell+1}\rho,\ldots,\alpha_{m+n}+\beta_{m+n}\rho)\neq\mathbf{0}\qquad AD-BC\neq 0\qquad C(\zeta_1+\theta_1\mathbf{i})+D\neq 0$$

$$(\zeta_1+\theta_1\mathbf{i},\widetilde{x}_1+\widetilde{y}_1\mathbf{i},\ldots,\widetilde{x}_\ell+\widetilde{y}_\ell\mathbf{i},x_{\ell+1}+y_{\ell+1}\mathbf{i},\ldots,x_m+y_m\mathbf{i},\zeta_2+\theta_2\mathbf{i},u_1+v_1\mathbf{i},\ldots,u_n+v_n\mathbf{i})\in\mathcal{Z}$$

$$(C(\zeta_1+\theta_1\mathbf{i})+D)\,(\zeta_2+\theta_2\mathbf{i})=A(\zeta_1+\theta_1\mathbf{i})+B$$

$$\sum_{p=1}^{\ell}(\alpha_p+\beta_p\rho)(\widetilde{x}_p+\widetilde{y}_p\mathbf{i})+\sum_{q=\ell+1}^{m}(\alpha_q+\beta_q\rho)(x_q+y_q\mathbf{i})$$
$$+(C(\zeta_1+\theta_1\mathbf{i})+D)\sum_{r=1}^{n}(\alpha_{m+r}+\beta_{m+r}\rho)(u_r+v_r\mathbf{i})=\xi_1+\xi_2(\zeta_1+\theta_1\mathbf{i})$$

where $\mathbf{i}$ is the imaginary unit. In particular, we consider for each $T\geq 1$

$$W^{\sim}(\mathbb{Q},T):=\{(\alpha_1,\ldots,v_n)\in W: H_1(\alpha_1,\ldots,\alpha_{m+n},\beta_1,\ldots,\beta_{m+n},A,B,C,D,\xi_1,\xi_2)\leq T\}$$

where we recall that $H_1(\alpha_1,\ldots,\xi_2)$ is finite if and only if $(\alpha_1,\ldots,\xi_2)\in\mathbb{Q}^{2m+2n+6}$.

Let π_1,π_2 be the projections on the first $2m+2n+6$ and the last $2m+2n+4$ coordinates, respectively.

Lemma 3.13. *For every $\varepsilon>0$, $\#\pi_2\left(W^{\sim}(\mathbb{Q},T)\right)\ll_\varepsilon T^\varepsilon$, for all $T\geq 1$.*

Proof. Fix $\varepsilon>0$ and let $c=c(W,\varepsilon)$ be the constant given by Proposition 3.11. Suppose also that $\#\pi_2\left(W^{\sim}(\mathbb{Q},T)\right)>cT^\varepsilon$ for some $T\geq 1$.

Then, by Proposition 3.11, there exists a continuous definable function $\delta:[0,1]\to W$ such that its restriction to $(0,1)$ is real analytic, $\delta_1=\pi_1\circ\delta:[0,1]\to\mathbb{R}^{2m+2n+6}$ is semi-algebraic and $\delta_2=\pi_2\circ\delta:[0,1]\to\mathbb{R}^{2m+2n+4}$ is non-constant. Thus, there exists an infinite connected $J\subseteq[0,1]$ such that $\delta_1(J)$ is contained in an algebraic curve and $\delta_2(J)$ has positive dimension.

Consider the coordinates

$$\alpha_1,\ldots,\alpha_{m+n},\beta_1,\ldots,\beta_{m+n},A,B,C,D,\xi_1,\xi_2,$$
$$\zeta_1,\theta_1,\widetilde{x}_1,\widetilde{y}_1,\ldots,\widetilde{x}_\ell,\widetilde{y}_\ell,x_{\ell+1},y_{\ell+1},\ldots,x_m,y_m,\zeta_2,\theta_2,u_1,v_1,\ldots,u_n,v_n$$

as functions on J and define

$$\tau_i=\zeta_i+\theta_i\mathbf{i},\qquad \widetilde{z}_p=\widetilde{x}_p+\widetilde{y}_p\mathbf{i},\qquad z_q=x_q+y_q\mathbf{i}\qquad w_r=u_r+v_r\mathbf{i}$$

with $i=1,2$, $p=1,\ldots,\ell$, $q=\ell+1,\ldots,m$ and $r=1,\ldots n$.

On J, the functions $\alpha_1,\ldots,\alpha_{m+n},\beta_1,\ldots,\beta_{m+n},A,B,C,D,\xi_1,\xi_2$ satisfy $2m+2n+6-1=2m+2n+5$ independent algebraic relations over $\mathbb{C}$ (because they are functions on an algebraic curve). Since $(\alpha_{\ell+1}+\beta_{\ell+1}\rho,\ldots,\alpha_{m+n}+\beta_{m+n}\rho)\neq\mathbf{0}$ and by the relations

$$\sum_{p=1}^{\ell}(\alpha_p+\beta_p\rho)\widetilde{z}_p+\sum_{q=\ell+1}^{m}(\alpha_q+\beta_q\rho)z_q+(C\tau_1+D)\sum_{r=1}^{n}(\alpha_{m+r}+\beta_{m+r}\rho)w_r=\xi_1+\xi_2\tau_1$$

$$(C\tau_1+D)\,\tau_2=A\tau_1+B$$

it follows that the $2m+2n+6+(m-\ell)+n=3m+3n+6-\ell$ functions

$$\alpha_1,\dots,\alpha_{m+n},\beta_1,\dots,\beta_{m+n},A,B,C,D,\xi_1,\xi_2,z_{\ell+1},\dots,z_m,w_1,\dots,w_n$$

satisfy $2m+2n+5+2=2m+2n+7$ independent algebraic relations over $F=\mathbb{C}(\tau_1,\tau_2)$. Finally, let

$$\mathcal{W}=(\tau_1(J),\widetilde{z}_1,\dots,\widetilde{z}_\ell,z_{\ell+1}(J),\dots,z_m(J),\tau_2(J),w_1(J),\dots,w_n(J))\subseteq\mathcal{Z},$$

which has positive dimension since $\delta_2(J)$ has positive dimension, and consider $\tau_1,z_{\ell+1},\dots,z_m,\tau_2,w_1,\dots,w_n$ as holomorphic functions on $\varphi(\mathcal{W})\subseteq\mathcal{C}(\mathbb{C})$. The algebraic relations found above can be analytically continued to an open disc D in $\varphi(\mathcal{W})\cap\widehat{\mathcal{C}}$. Therefore,

$$\operatorname{trdeg}_F F(z_{\ell+1},\dots,z_m,w_1,\dots,w_n)\le 3m+3n+6-\ell-(2m+2n+7)=m+n-\ell-1$$

which implies that $z_{\ell+1},\dots,z_m,w_1,\dots,w_n$ are algebraically dependent over F on D and thus, by Lemma 3.7, we get a contradiction, proving the proposition. □

Proof of Proposition 3.12. If $(\tau_1,\widetilde{z}_1,\dots,\widetilde{z}_\ell,z_{\ell+1},\dots,z_m,\tau_2,w_1,\dots,w_n)\in\mathcal{Z}(T)$, then there are integers $a_1,\dots,a_{m+n},b_1,\dots,b_{m+n},A,B,C,D$ with absolute value bounded by T and integers ξ_1,ξ_2 such that

$$(\alpha_{\ell+1}+\beta_{\ell+1}\rho,\dots,\alpha_{m+n}+\beta_{m+n}\rho)\neq\mathbf{0}\qquad AD-BC\neq0\qquad C\tau_1+D\neq0$$

$$(C\tau_1+D)\,\tau_2=A\tau_1+B$$

$$\sum_{i=1}^{\ell}(a_i+b_i\rho)\widetilde{z}_i+\sum_{i=\ell+1}^{m}(a_i+b_i\rho)z_i+(C\tau_1+D)\sum_{j=1}^{n}(a_{m+j}+b_{m+j}\rho)w_j=\xi_1+\xi_2\tau_1$$

And since $|\tau_1|,|\tau_2|,|A|,|B|,|C|,|D|,|a_1|,\dots,|a_{m+n}|,|b_1|,\dots,|b_{m+n}|\le T$ and $\widetilde{z}_p,z_q\in\mathcal{L}_{\tau_1}$, $w_r\in\mathcal{L}_{\tau_2}$ we have that

$$\begin{aligned}&\left|\sum_{p=1}^{\ell}(a_p+b_p\rho)\widetilde{z}_p+\sum_{q=\ell+1}^{m}(a_q+b_q\rho)z_q+(C\tau_1+D)\sum_{r=1}^{n}(a_{m+r}+b_{m+r}\rho)w_r\right|\\&\le\sum_{p=1}^{\ell}(|a_p|+|b_p|\,|\rho|)\,|\widetilde{z}_p|+\sum_{q=\ell+1}^{m}(|a_q|+|b_q|\,|\rho|)\,|z_q|+|C\tau_1+D|\sum_{r=1}^{n}(|a_{m+r}|+|b_{m+r}|\,|\rho|)\,|w_r|\\&\ll T\cdot\max\{1,|\tau_1|\}+(T\cdot\max\{1,|\tau_1|\})\cdot T\cdot\max\{1,|\tau_2|\}\ll T^4\end{aligned}$$

Therefore, we have

$$|\xi_1+\xi_2\tau_1|=\left|\sum_{p=1}^{\ell}(a_p+b_p\rho)\widetilde{z}_p+\sum_{q=\ell+1}^{m}(a_q+b_q\rho)z_q+(C\tau_1+D)\sum_{r=1}^{n}(a_{m+r}+b_{m+r}\rho)w_r\right|\ll T^4$$

from which we deduce that $|\xi_2| \ll T^5$, since $\mathrm{Im}(\tau_1) \geq \frac{1}{T}$ and

$$T^4 \gg |\xi_1 + \xi_2\tau_1| \geq |\mathrm{Im}(\xi_1 + \xi_2\tau_1)| = |\xi_2 \mathrm{Im}(\tau_1)| \geq \frac{|\xi_2|}{T}.$$

This implies that

$$|\xi_1| = |\xi_1 + \xi_2\tau_1 - \xi_2\tau_1| \leq |\xi_1 + \xi_2\tau_1| + |\xi_2| \cdot |\tau_1| \ll T^4 + T^5 \cdot T \ll T^6.$$

Hence we get that

$$\begin{aligned}&(a_1, \dots, a_{m+n}, b_1, \dots, b_{m+n}, A, B, C, D, \xi_1, \xi_2,\\ &\mathrm{Re}(\tau_1), \mathrm{Im}(\tau_1), \mathrm{Re}(\tilde{z}_1), \mathrm{Im}(\tilde{z}_1), \dots, \mathrm{Re}(\tilde{z}_\ell), \mathrm{Im}(\tilde{z}_\ell), \mathrm{Re}(z_{\ell+1}), \mathrm{Im}(z_{\ell+1}), \dots, \mathrm{Re}(z_m), \mathrm{Im}(z_m),\\ &\mathrm{Re}(\tau_2), \mathrm{Im}(\tau_2), \mathrm{Re}(w_1), \mathrm{Im}(w_1), \dots, \mathrm{Re}(w_n), \mathrm{Im}(w_n)) \in W^{\sim}(\mathbb{Q}, \nu T^6)\end{aligned}$$

for some positive constant ν. Finally, consider the map

$$\begin{aligned}\mathcal{Z}(T) &\longrightarrow \pi_2\left(W^{\sim}(\mathbb{Q}, \nu T^6)\right)\\ (\tau_1, \dots, w_n) &\longmapsto \left(\mathrm{Re}(\tau_1), \mathrm{Im}(\tau_1), \dots, \mathrm{Re}(w_n), \mathrm{Im}(w_n)\right).\end{aligned}$$

Since this map is injective, the conclusion follows from Lemma 3.13. □

3.5 Arithmetic bounds

Let $\mathcal{C}$ be as in Theorem 3.2 and let $\mathcal{C}'$ be the set of points $\mathbf{c} \in \widehat{\mathcal{C}}(\mathbb{C})$ such that there exists an isogeny $\phi_{\mathbf{c}} : E_{\mu(\mathbf{c})} \to E_{\lambda(\mathbf{c})}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^{m+n}$ with $(a_{\ell+1} + b_{\ell+1}\rho, \dots, a_{m+n} + b_{m+n}\rho) \neq \mathbf{0}$ and

$$\sum_{p=1}^{\ell}(a_p + b_p\rho)\widetilde{P}_p + \sum_{q=\ell+1}^{m}(a_q + b_q\rho)P_q(\mathbf{c}) + \sum_{r=1}^{n}(a_{m+r} + b_{m+r}\rho)\phi_{\mathbf{c}}(Q_r(\mathbf{c})) = O$$

where ρ is 0 in cases (i) and (ii), and a fixed generator for $\mathrm{End}(E_{\lambda_0})$ in case (iii). Moreover, we can also assume that $\phi_{\mathbf{c}}$ is a cyclic isogeny.

Since $\mathcal{C}$ is defined over $\overline{\mathbb{Q}}$, the curve $\widetilde{\mathcal{C}} = (J \circ \pi)(\mathcal{C})$ is also defined over $\overline{\mathbb{Q}}$ and thus, for every $\mathbf{c} \in \mathcal{C}'$, $(J(\lambda(\mathbf{c})), J(\mu(\mathbf{c}))) \in \widetilde{\mathcal{C}} \cap \bigcup_{N \geq 1} Y_0(N)$. As all the modular curves $Y_0(N)$ are defined over $\mathbb{Q}$, all the points $(J(\lambda(\mathbf{c})), J(\mu(\mathbf{c})))$ are algebraic, which implies that also $\lambda(\mathbf{c})$ and $\mu(\mathbf{c})$ are algebraic for every $\mathbf{c} \in \mathcal{C}'$. From this, it follows that $\mathcal{C}'$ is a subset of $\mathcal{C}(\overline{\mathbb{Q}})$ and thus we can define, for every $\mathbf{c}_0 \in \mathcal{C}'$, $D_0 := [k(\lambda(\mathbf{c}_0), \mu(\mathbf{c}_0)) : k]$, where k is the field of definition of $\mathcal{C}$.

All the constants appearing in this section depend only on $\mathcal{C}$, the field k and on the integers m, n, unless otherwise stated.

Lemma 3.14. *Let $\mathbf{c}_0 \in \mathcal{C}'$ and let N_0 be the minimal degree of an isogeny $\phi_{\mathbf{c}_0} : E_{\mu(\mathbf{c}_0)} \to E_{\lambda(\mathbf{c}_0)}$. Then, for every $\varepsilon > 0$, there exist positive constants γ_1, γ_2 (depending on ε) such that*

$$h(\lambda(\mathbf{c}_0)),\ h(\mu(\mathbf{c}_0)) \leq \gamma_1 D_0^{\varepsilon}$$

$$N_0 \leq \gamma_2 D_0^{2+\varepsilon}.$$

Proof. Fix $\mathbf{c}_0 \in \mathcal{C}'$, and let N_0 be as above. Note that an isogeny of minimal degree between two elliptic curves is necessarily cyclic, since otherwise it would factor as the composition of a cyclic isogeny and a multiplication-by-n map.

Therefore,

$$(J(\lambda(\mathbf{c}_0)), J(\mu(\mathbf{c}_0))) \in (\widetilde{\mathcal{C}} \cap Y_0(N_0))(\overline{\mathbb{Q}})$$

Since $\widetilde{\mathcal{C}}$ is asymmetric, we may apply [Hab10, Theorem 1.1] together with Proposition 2.11 to deduce

$$h(\lambda(\mathbf{c}_0)),\ h(\mu(\mathbf{c}_0)) \ll \log(1 + N_0).$$

Next, by Théorème 1.4 of [GR14b] , we have

$$N_0 \ll D_0^2 \cdot \max\left\{h_F(E_{\lambda(\mathbf{c}_0)}), \log(D_0), 1\right\}^2,$$

where $h_F(E_{\lambda(\mathbf{c}_0)})$ is the (stable) Faltings height of $E_{\lambda(\mathbf{c}_0)}$. By Proposition 2.1 of [Sil86], one has

$$h_F(E_{\lambda(\mathbf{c}_0)}) \ll h\left(j(E_{\lambda(\mathbf{c}_0)})\right) + 1.$$

Since $j(E_{\lambda(\mathbf{c}_0)})$ is a rational function in $\lambda(\mathbf{c}_0)$, Proposition 2.11 gives

$$h\left(j(E_{\lambda(\mathbf{c}_0)})\right) + 1 \ll h(\lambda(\mathbf{c}_0)) + 1 \ll \log(1 + N_0) \ll_{\epsilon_1} N_0^{\epsilon_1}$$

for every $\epsilon_1 > 0$. Moreover, for every $\epsilon_2 > 0$ we have $\log D_0 \ll_{\epsilon_2} D_0^{\epsilon_2}$. Hence

$$N_0 \ll D_0^2 \cdot \max\left\{h_F(E_{\lambda(\mathbf{c}_0)}), \log(D_0), 1\right\}^2 \ll_{\epsilon_1,\epsilon_2} D_0^{2+2\epsilon_2} \cdot N_0^{2\epsilon_1}.$$

Now fix $\varepsilon > 0$. Choosing $\epsilon_1 = \frac{\varepsilon}{8+4\varepsilon}$ and $\epsilon_2 = \frac{\varepsilon}{4}$ yields

$$N_0 \ll_{\varepsilon} D_0^{2+\varepsilon}.$$

Finally, recalling that $h(\lambda(\mathbf{c}_0)), h(\mu(\mathbf{c}_0)) \ll \log(1 + N_0) \ll_{\epsilon} N_0^{\epsilon}$, we conclude

$$h(\lambda(\mathbf{c}_0)),\ h(\mu(\mathbf{c}_0)) \ll_{\varepsilon} D_0^{\varepsilon}$$

for every $\varepsilon > 0$. □

Remark 3.15. Note that this lemma above is the only part of the proof where we need to use the hypothesis that $\mathcal{C}$ is asymmetric, while all the other steps are true also for non-asymmetric curves. Thus, if one was able to prove this lemma for an arbitrary $\mathcal{C}$ or any of the Conjectures 21.20, 21.23 or 21.24 from [Pil22], then Theorem 3.2 would follow for any $\mathcal{C}$.

Lemma 3.16. *Let $\mathbf{c}_0 \in \mathcal{C}'$. Then, there exist positive constants $\gamma_3, \gamma_4, \gamma_5$ such that*

$$\widehat{h}(P_q(\mathbf{c}_0)) \leq \gamma_3 D_0 \quad \textit{for every } q = \ell+1, \dots, m$$

$$\widehat{h}(\phi_{\mathbf{c}_0}(Q_r(\mathbf{c}_0))) \leq \gamma_4 D_0^4 \quad \textit{for every } r = 1, \dots, n.$$

Moreover, the $P_q(\mathbf{c}_0)$ and the $\phi_{\mathbf{c}_0}(Q_r(\mathbf{c}_0))$ are defined over a field $K \supseteq k(\lambda(\mathbf{c}_0), \mu(\mathbf{c}_0))$ with

$$[K : \mathbb{Q}] \leq \gamma_5 D_0^2.$$

Proof. We use the same notation as in the previous proof.

Using work of Zimmer [Zim76, Theorem], the previous lemma (with $\varepsilon = 1$) and the bounds (3.5), in case (i) we have

$$\widehat{h}(P_p(\mathbf{c}_0)) \leq h(P_p(\mathbf{c}_0)) + \gamma_6 \left(h(\lambda(\mathbf{c}_0)) + 1\right) \leq \gamma_7 \left(h(\lambda(\mathbf{c}_0)) + 1\right) \leq \gamma_2 D_0$$

while in case (ii) and (iii) we get the same estimate by

$$\widehat{h}(P_p(\mathbf{c}_0)) \leq h(P_p(\mathbf{c}_0)) + \gamma_6 \left(h(\mu(\mathbf{c}_0)) + 1\right) \leq \gamma_7 \left(h(\mu(\mathbf{c}_0)) + 1\right) \leq \gamma_2 D_0.$$

Similarly, $\widehat{h}(Q_r(\mathbf{c}_0)) \leq \gamma_8 D_0$. So, by Equation (3.4) and Lemma 3.14 (again with $\varepsilon = 1$), we get that

$$\widehat{h}(\phi_{\mathbf{c}_0}(Q_r(\mathbf{c}_0))) \leq \gamma_8 N_0 D_0 \leq \gamma_3 D_0^4$$

Lemma 7.2 in [BC16] implies that the $P_p(\mathbf{c}_0)$ and the $Q_r(\mathbf{c}_0)$ are defined over a field K_1 of degree $\leq \gamma_9 D_0$ over $\mathbb{Q}$. Moreover, by [MW90, Lemma 6.1], $\phi_{\mathbf{c}_0}$ is defined over a field K_2 of degree at most 12 over $k(\lambda(\mathbf{c}_0), \mu(\mathbf{c}_0))$, and thus $[K_2 : \mathbb{Q}] \leq 12 D_0$. Therefore, the points $\phi_{\mathbf{c}_0}(Q_r(\mathbf{c}_0))$ are defined over the compositum $K_1 K_2$ which has degree $\ll D_0^2$ over $\mathbb{Q}$. □

Next, we show that for any $\mathbf{c}_0 \in \mathcal{C}'$ we can choose "small" coefficients $a_i \in \mathbb{Z}$ for the linear relation.

Lemma 3.17. *For any $\mathbf{c}_0 \in \mathcal{C}'$, there exist $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^{m+n}$ with $(a_{\ell+1} + b_{\ell+1}\rho, \dots, a_{m+n} + b_{m+n}\rho) \neq \mathbf{0}$ and*

$$\sum_{p=1}^{\ell} (a_p + b_p \rho) \widetilde{P}_p + \sum_{q=\ell+1}^{m} (a_q + b_q \rho) P_q(\mathbf{c}_0) + \sum_{r=1}^{n} (a_{m+r} + b_{m+r}\rho) \phi_{\mathbf{c}_0}(Q_r(\mathbf{c}_0)) = O$$

and such that

$$\max\{|a_i|, |b_i|\} \leq \gamma_{10} D_0^{\eta_1}$$

for some positive constants γ_{10}, η_1.

Proof. For cases (i) and (ii), we already saw that we can take $\rho = 0$ and therefore we can choose $\mathbf{b} = \mathbf{0}$. So we can apply Lemma 6.1 of [BC16] (which is in turn based on a result by Masser [Mas88]), to the points $\widetilde{P}_p, P_q(\mathbf{c}_0)$ and $\phi_{\mathbf{c}_0}(Q_r(\mathbf{c}_0))$, using also Lemma 3.14 and the height bounds from the previous lemma.

In case (iii), we use again the above-mentioned lemma by Barroero and Capuano, this time with the points $\widetilde{P}_p, \rho\widetilde{P}_p, P_q(\mathbf{c}_0), \rho P_q(\mathbf{c}_0), \phi_{\mathbf{c}_0}(Q_r(\mathbf{c}_0))$ and $\rho\phi_{\mathbf{c}_0}(Q_r(\mathbf{c}_0))$, recalling that by (3.6), we have that

$$\widehat{h}(\rho P_q(\mathbf{c}_0)) \ll D_0^6 \cdot \widehat{h}(P_q(\mathbf{c}_0)) \ll D_0^7$$

$$\widehat{h}(\rho\phi_{\mathbf{c}_0}(Q_r(\mathbf{c}_0))) \ll D_0^6 \cdot \widehat{h}(\phi_{\mathbf{c}_0}(Q_r(\mathbf{c}_0))) \ll D_0^{10}.$$

A priori, applying [BC16, Lemma 6.1] in cases (ii) and (iii) gives $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^{m+n}$ with $(a_1 + b_1\rho, \ldots, a_{m+n} + b_{m+n}\rho) \neq \mathbf{0}$, but we claim that we cannot have $(a_{\ell+1} + b_{\ell+1}\rho, \ldots, a_{m+n} + b_{m+n}\rho) = \mathbf{0}$, otherwise it would mean that the constant points $\widetilde{P}_1, \ldots, \widetilde{P}_\ell$ are $\mathrm{End}(E_{\lambda_0})$-linearly dependent, contradicting our assumptions. □

For the next lemma, let $\tau_1(\mathbf{c}) = \tau_1(\varphi^{-1}_{|\mathcal{F}_\mathcal{B}}(\mathbf{c})) \in \mathcal{B}$ for every $\mathbf{c} \in \mathcal{C}(\mathbb{C})$ and similarly for $\tau_2(\mathbf{c})$, where φ is the uniformization map defined in Section 3.2.2.

Lemma 3.18. *There exist positive constants γ_{11}, γ_{12} such that for every $\mathbf{c}_0 \in \mathcal{C}'$ we have*

$$|\tau_1(\mathbf{c}_0)|, |\tau_2(\mathbf{c}_0)| \leq \gamma_{11} D_0^2$$

$$\mathrm{Im}(\tau_1(\mathbf{c}_0)), \mathrm{Im}(\tau_2(\mathbf{c}_0)) \geq \gamma_{12}\frac{1}{D_0^4}.$$

Proof. Let $\mathfrak{F}$ be the usual fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ (defined in (3.1)) and let $\tau \in \mathfrak{F}$. Then, Lemma 1 in [BMZ13] implies that $e^{2\pi\mathrm{Im}(\tau)} \leq 2079 + |j(\tau)|$. Hence, if $|j(\tau)| \leq 2$, then $\mathrm{Im}(\tau) \leq \frac{1}{2\pi}\log(2081) = \gamma_{13}$. Equivalently, for every $\tau \in \mathfrak{F}$ such that $\mathrm{Im}(\tau) > \gamma_{13}$, we have $|j(\tau)| > 2$. So, if $\mathrm{Im}(\tau) > \gamma_{13}$, we then get that

$$\mathrm{Im}(\tau) \leq \frac{1}{2\pi}\log(2079 + |j(\tau)|) \leq \frac{\log(2081)}{2\pi\log(2)}\log|j(\tau)|.$$

Therefore, for every $\tau \in \mathfrak{F}$ we have $\mathrm{Im}(\tau) \ll \max\{1, \log|j(\tau)|\}$.

Now, assume that $\tau_1(\mathbf{c}_0) = M \cdot \tau_1'$, for some $\tau_1' \in \mathfrak{F}$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then,

$$\mathrm{Im}(\tau_1(\mathbf{c}_0)) = \mathrm{Im}\left(\frac{a\tau_1' + b}{c\tau_1' + d}\right) = \frac{\mathrm{Im}(\tau_1')}{|c\tau_1' + d|^2} \leq \frac{\mathrm{Im}(\tau_1')}{c^2 - cd + d^2} \ll_M \max\{1, \log|j(\tau_1')|\}.$$

As the j-function is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, we have that

$$j(\tau_1(\mathbf{c}_0)) = j(M \cdot \tau_1') = j(\tau_1'),$$

so that $\mathrm{Im}(\tau_1(\mathbf{c}_0)) \ll_M \max\{1, \log|j(\tau_1(\mathbf{c}_0))|\}$.

Furthermore, we have that $j(\tau_1(\mathbf{c}_0)) = J(L(\tau_1(\mathbf{c}_0))) = J(\lambda(\mathbf{c}_0))$, where $J(\lambda) = 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$ and L was defined in (3.2). Since $\lambda(\mathbf{c}_0) \in \overline{\mathbb{Q}} \setminus \{0,1\}$, this implies that $j(\tau_1(\mathbf{c}_0)) \in \overline{\mathbb{Q}}$.

Then, using the inequality $\log|\alpha| \leq [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha)$ for every non-zero $\alpha \in \overline{\mathbb{Q}}$, we get

$$\begin{aligned} \log|j(\tau_1(\mathbf{c}_0))| &\leq [\mathbb{Q}(j(\tau_1(\mathbf{c}_0))) : \mathbb{Q}]\, h(j(\tau_1(\mathbf{c}_0))) = [\mathbb{Q}(J(\lambda(\mathbf{c}_0))) : \mathbb{Q}]\, h(J(\lambda(\mathbf{c}_0))) \\ &\ll [\mathbb{Q}(\lambda(\mathbf{c}_0)) : \mathbb{Q}]\,(h(\lambda(\mathbf{c}_0)) + 1) \ll D_0^2 \end{aligned}$$

by Lemma 3.14 and Proposition 2.11. Combining this with the previous bound gives $\mathrm{Im}(\tau_1(\mathbf{c}_0)) \leq \gamma_{14}(M) D_0^2$, for some positive constant $\gamma_{14}(M)$ depending on M. Since $\mathcal{B}$ is the union of finitely many translates of $\mathfrak{F}$ by elements of $\mathrm{SL}_2(\mathbb{Z})$, there are only finitely many such M to consider. Thus, we have that $\mathrm{Im}(\tau_1(\mathbf{c}_0)) \leq \gamma_{15} D_0^2$, where γ_{15} is an absolute constant, and that $|\mathrm{Re}(\tau_1(\mathbf{c}_0))| \ll 1$. Therefore, we get that $|\tau_1(\mathbf{c}_0)| \ll D_0^2$.

For the lower bound on the imaginary part, first note that if $\tau \in \mathfrak{F}$, then $\mathrm{Im}(\tau) \geq \frac{\sqrt{3}}{2}$. Again, assume that $\tau_1(\mathbf{c}_0) = M \cdot \tau_1'$, for some $\tau_1' \in \mathfrak{F}$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then,

$$\mathrm{Im}(\tau_1(\mathbf{c}_0)) = \mathrm{Im}\left(\frac{a\tau_1' + b}{c\tau_1' + d}\right) = \frac{\mathrm{Im}(\tau_1')}{|c\tau_1' + d|^2} \geq \frac{\mathrm{Im}(\tau_1')}{(|c| + |d|)^2 \cdot \max\{1, |\tau_1'|\}^2} \gg_M \frac{1}{\max\{1, |\tau_1'|\}^2}.$$

From before we get that $\mathrm{Im}(\tau_1') \ll \max\{1, \log|j(\tau_1')|\} = \max\{1, \log|j(\tau_1(\mathbf{c}_0))|\} \ll D_0^2$, which implies $|\tau_1'| \ll D_0^2$, since $\tau_1' \in \mathfrak{F}$ implies $|\mathrm{Re}(\tau_1')| \leq \frac{1}{2}$. Hence,

$$\mathrm{Im}(\tau_1(\mathbf{c}_0)) \gg_M \frac{1}{\max\{1, |\tau_1'|\}^2} \gg \frac{1}{D_0^4}.$$

As before, we need to consider only finitely many choices of $M \in \mathrm{SL}_2(\mathbb{Z})$, so we have that $\mathrm{Im}(\tau_1(\mathbf{c}_0)) \gg \frac{1}{D_0^4}$, where the implied constant is absolute.

Similar arguments give the respective bounds for $\tau_2(\mathbf{c}_0)$. $\square$

3.6 Proof of Theorem 3.2

We want to show that the set $\mathcal{C}'$ (defined at the start of the previous section) is finite. Since the map $\pi_{|\mathcal{C}} : \mathbf{c} \mapsto (\lambda(\mathbf{c}), \mu(\mathbf{c}))$ is finite-to-one, Northcott's theorem together with Lemma 3.14 reduces the problem to bounding the degree D_0 of $\lambda(\mathbf{c})$ and $\mu(\mathbf{c})$ over k.

Let $\mathbf{c}_0 \in \mathcal{C}'$ and $\sigma \in \mathrm{Gal}(\overline{k}/k)$. Notice that $\sigma(\mathbf{c}_0) \in \mathcal{C}'$. Indeed, we have

$$j\left(E_{\lambda(\sigma(\mathbf{c}_0))}\right) = j\left(E_{\sigma(\lambda(\mathbf{c}_0))}\right) = J\left(\sigma(\lambda(\mathbf{c}_0))\right) = \sigma\left(J\left(\lambda(\mathbf{c}_0)\right)\right) = \sigma\left(j\left(E_{\lambda(\mathbf{c}_0)}\right)\right)$$

and an analogous identity holds for $\mu(\mathbf{c}_0)$ in place of of $\lambda(\mathbf{c}_0)$. If N_0 is defined as in Lemma 3.14, it follows that

$$\Phi_{N_0}\left(j\left(E_{\lambda(\sigma(\mathbf{c}_0))}\right), j\left(E_{\mu(\sigma(\mathbf{c}_0))}\right)\right) = \sigma\left(\Phi_{N_0}\left(j\left(E_{\lambda(\mathbf{c}_0)}\right), j\left(E_{\mu(\mathbf{c}_0)}\right)\right)\right) = 0,$$

so there exists a cyclic isogeny $\phi_{\sigma(\mathbf{c}_0)} : E_{\mu(\sigma(\mathbf{c}_0))} \to E_{\lambda(\sigma(\mathbf{c}_0))}$ of degree N_0. Since $\deg \sigma(\phi_{\mathbf{c}_0}) = \deg \phi_{\mathbf{c}_0} = N_0$, we can take $\phi_{\sigma(\mathbf{c}_0)} = \sigma(\phi_{\mathbf{c}_0})$.

Thus, as $\mathcal{C}$ is defined over k, we also have

$$\sigma(\widetilde{P}_p) = \widetilde{P}_p\,, \qquad P_q(\sigma(\mathbf{c}_0)) = \sigma\left(P_q(\mathbf{c}_0)\right),$$

$$\phi_{\sigma(\mathbf{c}_0)}\left(Q_r(\sigma(\mathbf{c}_0))\right) = \phi_{\sigma(\mathbf{c}_0)}\left(\sigma\left(Q_r(\mathbf{c}_0)\right)\right) = \sigma\left(\phi_{\mathbf{c}_0}\left(Q_r(\mathbf{c}_0)\right)\right)$$

for every $p = 1, \ldots, \ell$, $q = \ell+1, \ldots, m$, $r = 1, \ldots, n$. Moreover, in case (iii), we can assume without loss of generality that the generator ρ of $\mathrm{End}(E_{\lambda_0})$ is defined over k, so that $\sigma(\rho) = \rho$. Recall that we are using [Sil94, Proposition II.1.1] to identify endomorphisms with complex numbers. Furthermore, [Sil94, Proposition II.2.2] guarantees that under this identification the action of $\mathrm{Gal}(\overline{k}/k)$ on these two objects is the same.

So, in all cases, we have:

$$\begin{aligned}&\sum_{p=1}^{\ell}(a_p + b_p\rho)\widetilde{P}_p + \sum_{q=\ell+1}^{m}(a_q + b_q\rho)P_q(\sigma(\mathbf{c}_0)) + \sum_{r=1}^{n}(a_{m+r} + b_{m+r}\rho)\phi_{\sigma(\mathbf{c}_0)}(Q_q(\rho(\mathbf{c}_0)))\\ &= \sigma\Big(\sum_{p=1}^{\ell}(a_p + b_p\rho)\widetilde{P}_p + \sum_{q=\ell+1}^{m}(a_q + b_q\rho)P_q(\mathbf{c}_0) + \sum_{r=1}^{n}(a_{m+r} + b_{m+r}\rho)\phi_{\mathbf{c}_0}(Q_r(\mathbf{c}_0))\Big) = O\end{aligned}$$

on $E_{\lambda(\sigma(\mathbf{c}_0))}$, since the a_i and b_i are integers.

Now, consider the point $\varphi^{-1}_{|\mathcal{F}_{\mathcal{B}}}(\sigma(\mathbf{c}_0)) \in \mathcal{Z}$ with coordinates

$$(\tau_1^\sigma, \widetilde{z}_1, \ldots, \widetilde{z}_\ell, z_{\ell+1}^\sigma, \ldots, z_m^\sigma, \tau_2^\sigma, w_1^\sigma, \ldots, w_n^\sigma)$$

(here the superscript σ does not denote a Galois conjugate). By the previous equation and lemmas 3.17 and 3.18 we have relations

$$\sum_{p=1}^{\ell}(a_p + b_p\rho)\widetilde{z}_p + \sum_{q=\ell+1}^{m}(a_q + b_q\rho)z_q^\sigma + (C^\sigma\tau_1^\sigma + D^\sigma)\left(\sum_{r=1}^{n}(a_{m+r} + b_{m+r}\rho)w_r^\sigma\right) \in \mathbb{Z} + \mathbb{Z}\tau_1^\sigma,$$

$$\tau_2^\sigma = \frac{A^\sigma\tau_1^\sigma + B^\sigma}{C^\sigma\tau_1^\sigma + D^\sigma}$$

with $(a_{\ell+1} + b_{\ell+1}\rho, \ldots, a_{m+n} + b_{m+n}\rho) \neq \mathbf{0}$ and

$$\max\{|a_i|, |b_i|\} \leq \gamma_{10}D_0^{\eta_1}, \qquad |\tau_1^\sigma|, |\tau_2^\sigma| \leq \gamma_{11}D_0^2, \qquad \mathrm{Im}(\tau_1^\sigma) \geq \gamma_{12}\frac{1}{D_0^4}$$

and $|A^\sigma|, |B^\sigma|, |C^\sigma|, |D^\sigma| \leq \gamma_{16} N_0 \leq \gamma_{17} D_0^3$ by Lemma 3.6 and Lemma 3.14.

So,

$$\varphi^{-1}_{|_{\mathcal{F}_\mathcal{B}}}(\sigma(\mathbf{c}_0)) \in \mathcal{Z}(\gamma D_0^\eta)$$

where $\gamma = \max\left\{\gamma_{10}, \gamma_{11}, \frac{1}{\gamma_{12}}, \gamma_{17}\right\}$ and $\eta = \max\{\eta_1, 4\}$.

There are at least $[k(\mathbf{c}_0) : k] \geq [k(\lambda(\mathbf{c}_0), \mu(\mathbf{c}_0)) : k] = D_0$ different

$$\left(\tau_1^\sigma, \widetilde{z}_1, \ldots, \widetilde{z}_\ell, z_{\ell+1}^\sigma, \ldots, z_m^\sigma, \tau_2^\sigma, w_1^\sigma, \ldots, w_n^\sigma\right)$$

in $\mathcal{Z}(\gamma D_0^\eta)$. However, applying Proposition 3.12 with $\varepsilon = \frac{1}{2\eta}$ gives a contradiction if D_0 is large enough. This proves that D_0 is bounded and, consequently, Theorem 3.2.

Chapter 4

Unlikely intersections in families of abelian varieties

4.1 Introduction

Let S be a smooth, irreducible, quasi-projective curve, and let $\pi : \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$, both defined over $\overline{\mathbb{Q}}$. For any (not necessarily closed) point $s \in S$ we denote the fiber of $\mathcal{A}$ over s by $\mathcal{A}_s$. Let $O : S \to \mathcal{A}$ be the zero section of $\mathcal{A}$ and consider an irreducible curve $\mathcal{C} \subseteq \mathcal{A}$, also defined over $\overline{\mathbb{Q}}$.

Recall that an irreducible component of a subgroup scheme of $\mathcal{A}$ is either a component of an algebraic subgroup of a fiber or it dominates the base curve S. We say that a subgroup scheme is *flat* if all of its irreducible components are of the latter kind.

We call $\mathcal{A} \to S$ *isotrivial* if it becomes constant after a base change, i.e. $\mathcal{A} \times_S S' \cong A \times_{\overline{\mathbb{Q}}} S'$ for some finite base change $S' \to S$ and some fixed abelian variety $A/\overline{\mathbb{Q}}$. Let $A_0 \times S$ be the largest constant abelian subscheme of $\mathcal{A} \to S$, we say that a section $\sigma : S \to \mathcal{A}$ is constant if there exists $a_0 \in A_0(\mathbb{C})$ such that σ is the composition of $S \to A_0 \times S$, $s \mapsto (a_0, s)$ with the inclusion of $A_0 \times S$ into $\mathcal{A}$.

The goal of this chapter is to further investigate the intersections of $\mathcal{C}$ with subgroup schemes of $\mathcal{A}$. In [BC20], Barroero and Capuano studied the intersections of $\mathcal{C}$ with flat subgroup schemes of codimension at least 2 and proved that if $\mathcal{C}$ is not contained in a proper subgroup scheme, then its intersection with the union of all such codimension ≥ 2 flat subgroup schemes of $\mathcal{A}$ is finite.

In the isotrivial case or if $\mathcal{C}$ is contained in a fixed fiber, this has already been addressed by Habegger and Pila [HP16, Theorem 9.14], who proved the Zilber–Pink conjecture for curves in abelian varieties defined over $\overline{\mathbb{Q}}$. Thus, our focus is instead on the case where the abelian scheme $\mathcal{A} \to S$ is not isotrivial and $\mathcal{C}$ is not contained in a fixed fiber.

In this chapter, we extend these results by considering the intersections of $\mathcal{C}$ with the proper algebraic subgroups of the CM fibers of $\mathcal{A}$, proving the following theorem.

Theorem 4.1. *Let S and $\mathcal{A} \to S$ be as above and assume that $\mathcal{A}$ is not isotrivial. Let $\mathcal{C} \subseteq \mathcal{A}$ an irreducible curve defined over $\overline{\mathbb{Q}}$ that is neither contained in a fixed fiber nor in a translate of a proper flat subgroup scheme of $\mathcal{A}$ by a constant section, even after a finite base change. Then, the intersection of $\mathcal{C}$ with the union of all proper algebraic subgroups of the CM fibers of $\mathcal{A}$ is a finite set.*

Since every algebraic subgroup of an abelian variety is a union of irreducible components of the kernel of an endomorphism, the theorem can be restated as follows: under the same assumptions as above, there are at most finitely many $P \in \mathcal{C}(\mathbb{C})$ such that $\mathcal{A}_{\pi(P)}$ has complex multiplication and there exists a non-zero $f \in \mathrm{End}(\mathcal{A}_{\pi(P)})$ such that $f(P) = O_{\pi(P)}$.

In [Bar19], Barroero proved the same result in the case of a fibered power of an elliptic scheme. Thus, Theorem 4.1 can be viewed as a generalization of Barroero's result to more general abelian schemes.

The above theorem also proves a stronger partial version of Conjecture 6.1 of [Pin05b], since Pink's conjecture only considers algebraic subgroups of codimension at least 2 of the fibers. As a matter of fact, Theorem 4.1 is a particular case of the Zilber–Pink conjecture for a curve in an abelian scheme, which is known to imply Conjecture 6.1 of [Pin05b] for abelian schemes.

Our proof of Theorem 4.1 follows the well-established Pila-Zannier strategy, first introduced in [PZ08] and later used, among others, by Masser and Zannier [MZ10, MZ12], by Barroero and Capuano [BC16, Bar19, BC17, BC20] and in the previous chapter.

To implement this strategy, we first reduce the problem to the case of restrictions of the universal family of abelian varieties over a quasi-projective curve in the moduli space $\mathbb{A}_g$ of principally polarized abelian varieties of dimension g. Using a result of Peterzil and Starchenko, after restricting to a suitable fundamental domain, the uniformizing map of the universal family is definable in the o-minimal structure $\mathbb{R}_{\text{an, exp}}$. Consequently, the preimage of $\mathcal{C}$ under this map is a definable surface X.
Let $\mathcal{C}'$ be the subset of $\mathcal{C}$ we want to prove to be finite. Then, each point $P_0 \in \mathcal{C}'$ correspond to a point on X lying on a subvariety defined by equations with integer coefficients. We then use a result by Habegger and Pila, which states that the number of points on X lying on such subvarieties with coefficients bounded in absolute value by T is at most $\ll T^{\varepsilon}$, provided that the abelian logarithm of the generic point of $\mathcal{C}$ generates a field of sufficiently large transcendence degree over the field generated by the period matrix.

We then use a result by Barroero and Capuano, based on an earlier result by Masser [Mas88], to construct a linear combination of a specific basis of endomorphisms of $\mathcal{A}_{\pi(P_0)}$, with coefficients bounded by a constant times a positive power of $[\mathbb{Q}(P_0) : \mathbb{Q}]$ and such that P_0 lies in the kernel of this linear combination. In order to do this, we use the bounds for the canonical height developed in Section 2.6. Furthermore, since all Galois conjugates of P_0 remain in $\mathcal{C}'$, there are at least D_0 points on X lying on subvarieties whose coefficients are bounded in absolute value by some positive power of D_0. Together with the previous estimate, this implies that D_0 is uniformly bounded. By Northcott's theorem, this establishes the claim of the theorem.

Remark 4.2. Before proceeding, we note that if $S \subseteq \mathbb{A}_g$ is not a special curve (as explained in Section 4.3, we may always assume $S \subseteq \mathbb{A}_g$), then the André-Oort conjecture for $\mathbb{A}_g$ (proved by Tsimerman [Tsi18]) guarantees that only finitely many points $s \in S(\mathbb{C})$ correspond to CM fibers $\mathcal{A}_s$, which in turn implies Theorem 4.1. Hence, one may assume that $S = \pi(\mathcal{C})$ is a Shimura curve, though this assumption will not be used in the rest of the chapter.

Remark 4.3. Observe that the Zilber–Pink conjecture would imply Theorem 4.1 even when $\mathcal{C}$ is contained in a translate of a proper flat subgroup scheme of $\mathcal{A}$ by a non-torsion section. Unfortunately, the functional transcendence results used in this chapter only allow us to prove the theorem in the form stated above.

We use Vinogradov's $\ll$ notation: for real-valued functions $f(T)$ and $g(T)$, we write $f(T) \ll g(T)$ if there exists a constant $\gamma > 0$ such that $f(T) \leq \gamma g(T)$ for all sufficiently large T. When not explicitly stated, the implied constant is either absolute or depends only on $S, \mathcal{A}, g, \mathcal{C}$ and other fixed data. We use subscripts to indicate any additional dependence of the implied constant.

4.2 Preliminaries

For the basic results about abelian varieties we refer to Section 2.5.1.

4.2.1 Moduli spaces, universal families and their uniformizations

Let $g, n \geq 1$ be positive integers and $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_g)$, with d_i positive integers such that d_i divides d_{i+1} for every $i = 1, \ldots, g-1$. We define $\mathbb{A}_{g,\mathbf{D},n}$ as the moduli space of complex abelian abelian varieties of dimension g, polarization type $\mathbf{D}$ and with principal level-n-structure. For each type $\mathbf{D}$ and $n \geq 3$, the moduli space $\mathbb{A}_{g,\mathbf{D},n}$ is a fine moduli space [MFK94, Theorem 7.9]. In other words, there is a universal family $\pi : \mathfrak{A}_{g,\mathbf{D},n} \to \mathbb{A}_{g,\mathbf{D},n}$, which, like $\mathbb{A}_{g,\mathbf{D},n}$, is defined over $\overline{\mathbb{Q}}$. For the rest of the chapter we will consider $\mathfrak{A}_{g,\mathbf{D},n}$ and $\mathbb{A}_{g,\mathbf{D},n}$ as irreducible quasi-projective varieties.

It is well-known (see for example Chapter 8 of [BL04]) that $\mathbb{A}^{an}_{g,\mathbf{D},n}$, the analytification of $\mathbb{A}_{g,\mathbf{D},n}$, can be realized as a quotient of $\mathbb{H}_g$ by a suitable finite index subgroup $\Gamma_{\mathbf{D},n}$ of $\mathrm{Sp}_{2g}(\mathbb{Z})$, where

$$\mathbb{H}_g := \left\{ Z \in \mathrm{Mat}_g(\mathbb{C}) : Z = Z^t,\, \mathrm{Im}(Z) > 0 \right\}$$

and $\mathrm{Sp}_{2g}(\mathbb{Z}) := \{M \in \mathrm{Mat}_{2g}(\mathbb{Z}) : M^t J M = J\}$ (here $J := \left(\begin{smallmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{smallmatrix}\right)$) acts on $\mathbb{H}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

Remark 4.4. We will show in Section 4.3 that we can always reduce the problem to studying principally polarized abelian varieties. Moreover, the choice of the level structure is not important for our proof of Theorem 4.1. So, for the rest of the chapter, we fix $\mathbf{D} = \mathbf{1}_g$ and $n = 3$ and omit those indices from the notation when they are clear from the context.

Note that $\mathbb{H}_g$ is an open subset, in the Euclidean topology, of

$$\left\{M \in \mathrm{Mat}_g(\mathbb{C}) : M = M^t\right\} \cong \mathbb{C}^{\frac{g(g+1)}{2}}$$

and that we can see $\mathbb{H}_g$ as a semialgebraic subset of $\mathbb{R}^{2g^2}$, by identifying a complex number with its real and imaginary parts. Furthermore, the quotient map $u_b : \mathbb{H}_g \to \mathbb{A}_g^{an}$ is holomorphic.

Similarly, we have an holomorphic uniformization map for the universal family, given by theta functions, $u : \mathbb{H}_g \times \mathbb{C}^g \to \mathfrak{A}_g^{an}$, such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{H}_g \times \mathbb{C}^g & \xrightarrow{\ u\ } & \mathfrak{A}_g^{an} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{H}_g & \xrightarrow[u_b]{} & \mathbb{A}_g^{an} \end{array}$$

Now, we would like to find a subset of $\mathbb{H}_g \times \mathbb{C}^g$ over which u is invertible.

By [Igu72, Section V.4], there is a semialgebraic set $\mathfrak{F}_g$ of $\mathbb{H}_g$ which can be used as a fundamental domain for the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathbb{H}_g$. If Γ is a finite index subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$ and $\sigma_1 = \mathbf{1}_{2g}, \sigma_2, \ldots, \sigma_n$ is a complete set of representatives of its right cosets, then

$$\mathfrak{F}_\Gamma := \bigcup_{i=1}^{n} \sigma_i \cdot \mathfrak{F}_g \tag{4.1}$$

is called a *Siegel fundamental domain for* Γ and can be used as a fundamental domain for the action of Γ on $\mathbb{H}_g$.

For a fixed $\tau \in \mathbb{H}_g$ we have a principally polarized abelian variety $A_\tau = \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$. In this case, let $L_\tau := \{z \in \mathbb{C}^g : z = u + \tau v \text{ with } u, v \in [0,1)^g\}$ be the fundamental parallelogram for the lattice $\mathbb{Z}^g + \tau\mathbb{Z}^g$. Moreover, let $\Gamma = \Gamma_{\mathbf{D},n}$ as above and define

$$\mathcal{F}_g := \{(\tau, z) \in \mathbb{H}_g \times \mathbb{C}^g : \tau \in \mathfrak{F}_\Gamma, z \in L_\tau\}.$$

Then, the restriction of u to $\mathcal{F}_g$ is finite-to-one. Consider a curve $\mathcal{C} \subseteq \mathfrak{A}_g$ as in Theorem 4.1 and set

$$\mathcal{Z} = u^{-1}(\mathcal{C}(\mathbb{C})) \cap \mathcal{F}_g. \tag{4.2}$$

Finally, let $S \subseteq \mathbb{A}_g$ be a smooth, irreducible, locally closed curve and let $\mathcal{A} = \mathfrak{A}_g \times_{\mathbb{A}_g} S \to S$. Define the constant part (or $\overline{\overline{\mathbb{Q}}(S)}/\overline{\mathbb{Q}}$-trace) of $\mathcal{A} \to S$ as the largest abelian subvariety A_0 of the generic fiber $\mathcal{A}_\eta$ which can be defined over $\overline{\mathbb{Q}}$ (see also [Lan83a, Section VIII.3] for more details).

Let D be an open disc on $\mathcal{C}(\mathbb{C})$ and consider τ and z as holomorphic functions on D. The following functional transcendence result is a consequence of Theorem 7.1 of [Dil21] (which is in turn based on a result by Gao [Gao20]).

Lemma 4.5. *Let S, $\mathcal{A}$, $\mathcal{C}$ and D as above and let $F = \mathbb{C}(\tau)$. Under the assumptions of Theorem 4.1, we have* $\mathrm{tr.deg.}_F F(z) = g$ *on D.*

Proof. By contradiction, assume that $\mathrm{tr.deg.}_F F(z) < g$. Then Theorem 7.1 of [Dil21] implies the existence of a proper subvariety $\mathcal{W}$ of $\mathcal{A}$, containing $\mathcal{C}$ and such that, over $\overline{\overline{\mathbb{Q}}(S)}$, every irreducible component of $\mathcal{W}_\eta$ is a translate of an abelian subvariety of $\mathcal{A}_\eta$ by a point in $(\mathcal{A}_\eta)_{\mathrm{tors}} + A_0(\overline{\mathbb{Q}})$. This means that, up to finite base change, $\mathcal{C}$ is contained in a translate of a proper subgroup scheme by a point in $A_0(\overline{\mathbb{Q}})$, contradicting the hypotheses on $\mathcal{C}$ in Theorem 4.1. □

4.2.2 Heights

Let h denote the logarithmic absolute Weil height on $\mathbb{P}^N$, as defined in Chapter 2 and, if α is an algebraic number, define $h(\alpha) = h([1 : \alpha])$. Define also the multiplicative Weil height as $H(P) = \exp(h(P))$. More generally, if V is a projective variety and D is a divisor, denote by $h_{V,D}$ a Weil height on V associated to D (see Section 2.4).

For an abelian variety A defined over a number field and a divisor D, we also have the Néron–Tate height $\widehat{h}_{A,D}$, defined as in Theorem 2.32. We also denote by $h_F(A)$ the stable Faltings height of A (see [Fal83]), assuming that A has semistable reduction everywhere. This assumption can always be ensured by passing to a suitable field extension.

Finally, we will also need another definition of height (from [HP12, Section 7]), which generalizes the height defined in Definition 3.8.

Definition 4.6. If $d \in \mathbb{Z}_{\geq 1}$ and α is a complex number, we define the d-height of α as

$$H_d(\alpha) := \min \left\{ H([a_0 : \ldots : a_d]) : [a_0 : \ldots : a_d] \in \mathbb{P}^d(\mathbb{Q}) \text{ s.t. } a_0 + a_1\alpha + \ldots + a_d\alpha^d = 0 \right\}$$

where we use the convention $\min \emptyset = +\infty$. For $(\alpha_1, \ldots, \alpha_N) \in \mathbb{C}^N$, we also define $H_d(\alpha_1, \ldots, \alpha_N) = \max \{H_d(\alpha_i)\}$.

Note that $H_d(\alpha_1, \ldots, \alpha_N)$ is finite if and only if $\alpha_1, \ldots, \alpha_N$ are all algebraic numbers of degree at most d.

Lemma 4.7. *For any $\alpha \in \overline{\mathbb{Q}}$ of degree at most d we have*

$$H_d(\alpha) \leq 2^d H(\alpha)^d \qquad \textit{and} \qquad |\alpha| \leq \sqrt{d+1} \cdot H_d(\alpha).$$

Proof. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n \in \mathbb{Q}[x]$ be a polynomial of degree $n \leq d$ such that $f(\alpha) = 0$ and let $m_\alpha(x) \in \mathbb{Z}[x]$ be the minimal polynomial of α (so its coefficients are coprime). Since in the definition of $H_d(\alpha)$ we are considering the coefficients of f as a point in a projective space, we may assume that the coefficients of f are integers with $\gcd(a_0, \ldots, a_n) = 1$.

For every $p \in \mathbb{C}[x]$ let $M(p)$ denote the Mahler measure of p, as in [BG06, Section 1.6]. By [BG06, Proposition 1.6.6],

$$M(m_\alpha) = H(\alpha)^{[\mathbb{Q}(\alpha):\mathbb{Q}]} \leq H(\alpha)^d.$$

Moreover, [BG06, Lemma 1.6.7] gives

$$\|f\|_\infty := \max\{|a_0|, \ldots, |a_n|\} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} M(f) \leq 2^n M(f) \leq 2^d M(f).$$

Since the coefficients of f are coprime integers, $H([a_0 : \ldots : a_n]) = \|f\|_\infty$. Hence

$$\begin{aligned} H_d(\alpha) &= \min\left\{\|f\|_\infty : f \in \mathbb{Z}[x] \text{ with coprime coefficients s.t. } \deg(f) \leq d \text{ and } f(\alpha) = 0\right\} \\ &\leq \|m_\alpha\|_\infty \leq 2^d M(m_\alpha) \leq 2^d H(\alpha)^d \end{aligned}$$

which proves the first inequality.

For the second inequality, note first that [BG06, Proposition 1.6.6] implies $|\alpha| \leq M(f)$ for any $f \in \mathbb{Z}[x]$ such that $f(\alpha) = 0$. Furthermore, by [BG06, Lemma 1.6.7], we also have that $M(f) \leq \sqrt{\deg(f) + 1} \cdot \|f\|_\infty$. Taking the minimum over all polynomials $f \in \mathbb{Z}[x]$ with coprime coefficients and $\deg(f) \leq d$ such that $f(\alpha) = 0$ then yields the desired bound $|\alpha| \leq \sqrt{d+1} \cdot H_d(\alpha)$. □

4.2.3 Complex Multiplication

In this section, we recall the basic definitions and key facts about complex multiplication for abelian varieties defined over fields of characteristic 0, which will be used throughout this chapter. For further details on this topic, we refer to [Lan83b, Shi98, Mil20].

Definition 4.8. A *CM field* K is a totally imaginary quadratic extension of a totally real number field. That is, K has the form $K = K_0(\sqrt{\alpha})$, where K_0 is a totally real field, i.e., a number field whose embeddings into $\mathbb{C}$ are all real, and $\alpha \in K_0$ satisfies the condition that each embedding of K_0 into $\mathbb{C}$ maps α to a negative real number.

Definition 4.9. An abelian variety A of dimension g is said to have *Complex Multiplication* (CM) if its endomorphism algebra $\mathrm{End}^0(A)$ contains a commutative semisimple subalgebra of degree $2g$ over $\mathbb{Q}$. We say that A has CM by the CM field K (of degree $2g$) if there exists an embedding $K \hookrightarrow \mathrm{End}^0(A)$.

Note that a simple abelian variety A has complex multiplication if and only if $\mathrm{End}^0(A)$ is a CM field of degree $2\dim(A)$. In general, an abelian variety has complex multiplication if and only if each of its simple factors up to isogeny has complex multiplication.

If A is a simple CM abelian variety of dimension g, then $\mathrm{End}^0(A) \cong K$ is a CM field and there is a set $\Phi = \{\phi_1, \dots, \phi_g\}$ of complex embeddings of K such that $\Phi \cup \overline{\Phi}$ is the set of all complex embeddings of K and $T_O(A) \cong \prod_{i=1}^{g} \mathbb{C}_{\phi_i}$, where $\mathbb{C}_{\phi_i}$ is a 1-dimensional $\mathbb{C}$-vector space on which $\alpha \in K$ acts as $\phi_i(\alpha)$. We call the pair (K, Φ) a *CM-type* of A. In particular, by Proposition 3.13 of [Mil20], (K, Φ) is primitive, i.e. it is not induced by a CM-type of a proper CM subfield of K.

4.2.4 O-minimality and definable sets

In this section we will use the properties and results about o-minimal structures introduced in Section 3.3. For more details, we refer again to [vdD98] and [vdDM96].

For this chapter we will need a more general version of Proposition 3.11. For a family $Z \subseteq \mathbb{R}^M \times \mathbb{R}^N = \mathbb{R}^{M+N}$, a positive integer d and a positive real number T define

$$Z^{\sim}(d, T) := \{(y, z) \in Z : H_d(y) \leq T\}$$

where $H_d(y)$ is the d-height given by Definition 4.6. Let also π_1, π_2 be the projections of Z to the first M and last N coordinates, respectively.

Proposition 4.10 ([HP16], Corollary 7.2)**.** *Let $Z \subseteq \mathbb{R}^{M+N}$ be a definable set. For every positive integer d and every $\varepsilon > 0$ there exists a positive constant $c = c(Z, d, \varepsilon)$ with the following property. If $T \geq 1$ and $|\pi_2(Z^{\sim}(d, T))| > cT^{\varepsilon}$, then there exists a continuous definable function $\delta : [0, 1] \to Z$ such that:*

1. *the restriction $\delta|_{(0,1)}$ is real analytic (since $\mathbb{R}_{\text{an, exp}}$ admits analytic cell decomposition);*
2. *the composition $\pi_1 \circ \delta : [0, 1] \to \mathbb{R}^M$ is semi-algebraic and its restriction to $(0, 1)$ is real analytic;*
3. *the composition $\pi_2 \circ \delta : [0, 1] \to \mathbb{R}^N$ is non-constant.*

We conclude this section by showing that the set $\mathcal{Z}$ defined in (4.2) is definable in $\mathbb{R}_{\text{an,exp}}$.

From now on, we use the term "definable" to mean definable in $\mathbb{R}_{\text{an,exp}}$. A complex set or function is said to be definable if it is definable as a real object, considering its real and imaginary parts separately. We may assume that $\mathfrak{A}_g$ is embedded in some projective space. By Theorem 1.2 of [PS13], there is an open subset U of $\mathbb{H}_g \times \mathbb{C}^g$ containing $\mathcal{F}_g$ such that the restriction of the uniformizing map u to U is definable. Since $\mathcal{F}_g$ is a semialgebraic subset of $\mathbb{H}_g \times \mathbb{C}^g$, it follows that u is definable when restricted to $\mathcal{F}_g$. Consequently, as $\mathcal{C}$ is semi-algebraic, we conclude that $\mathcal{Z}$ is definable.

4.3 Reduction to the universal family of principally polarized abelian varieties

In this section, we reduce to the case where $\mathcal{A} = \mathfrak{A}_g \times_{\mathbb{A}_g} S$, with $S \subseteq \mathbb{A}_g$ a smooth, irreducible, locally closed curve defined over $\overline{\mathbb{Q}}$. The results of this section are inspired by Section 2 of [BC20].

The first result of this section allows us to perform finite base changes.

Lemma 4.11. *Let $\mathcal{C}$ be as in Theorem 4.1. Let $\ell : S' \to S$ be a finite étale cover and $\mathcal{A}' = \mathcal{A} \times_S S'$. Let also $\rho : \mathcal{A}' \to \mathcal{A}$ be the projection map. Then, if the claim of Theorem 4.1 holds for all irreducible components of $\rho^{-1}(\mathcal{C})$, then it holds for $\mathcal{C}$.*

Proof. By the proof of Lemma 2.1 of [BC20] we have that ρ is flat and finite. By [Har77, Corollary III.9.6], we have that if $X \subseteq \mathcal{A}$ is an irreducible variety, then the dimension of each irreducible component of $\rho^{-1}(X)$ is equal to $\dim X$. Moreover, if X dominates S, then every irreducible component of $\rho^{-1}(X)$ dominates S'. In particular, this shows that the preimages of the flat subgroup schemes of $\mathcal{A}$ are flat subgroup schemes of $\mathcal{A}'$ of the same dimension. This implies that if $\mathcal{C}$ satisfies the hypotheses of Theorem 4.1, then the same is true for each irreducible component of $\rho^{-1}(\mathcal{C})$. Finally, the preimages of any point of $\mathcal{C}$ lying in a proper algebraic subgroup of a CM fiber $\mathcal{A}_s$, where $s \in S(\mathbb{C})$, are contained in proper algebraic subgroups of fibers of $\mathcal{A}'$, which are still CM, since for $s' \in S'(\mathbb{C})$ and $s \in S(\mathbb{C})$ such that $\ell(s') = s$, then $\mathcal{A}_s \cong \mathcal{A}'_{s'}$. □

Next, let $\mathcal{A}$ and $\mathcal{A}'$ be abelian schemes over the same curve S and let $f_\eta : \mathcal{A}'_\eta \to \mathcal{A}_\eta$ be an isogeny between the generic fibers defined over $\overline{\mathbb{Q}}(S)$. Then, f_η extends to an isogeny $f : \mathcal{A}' \to \mathcal{A}$ between the abelian schemes (see the proof of Lemma 2.2 of [BC20]).

Lemma 4.12. *Let $\mathcal{A}, \mathcal{A}', f_\eta$ and f as above and $\mathcal{C}$ as in Theorem 4.1. Then, if the claim of Theorem 4.1 holds for all irreducible components of $f^{-1}(\mathcal{C})$, then it holds for $\mathcal{C}$.*

Proof. For every $s \in S$, the map $f_s : \mathcal{A}'_s \to \mathcal{A}_s$ is an isogeny. In particular, the images and preimages of algebraic subgroups under f_s remain algebraic subgroups, and dimensions are preserved. Moreover, since isogenous abelian varieties have isomorphic endomorphism algebras, it follows that $\mathcal{A}_s$ is CM if and only if $\mathcal{A}'_s$ is CM. Now, consider the preimage under f of any intersection of $\mathcal{C}$ with the union of the proper algebraic subgroups of the CM fibers of $\mathcal{A}$. Since this preimage lies in a proper algebraic subgroup of a CM fiber of $\mathcal{A}'$, and by assumption the claim of Theorem 4.1 holds for all irreducible components of $f^{-1}(\mathcal{C})$, we conclude that the set of such points is finite. This proves the result. □

Now, as S is irreducible, smooth and quasi-projective, by [GW23, Theorem 27.291], we can take a relatively ample line bundle $\mathcal{L}$ on $\mathcal{A} \to S$. This line bundle induces a polarization on $\mathcal{A} \to S$ of type $D = (d_1, \dots, d_g)$. By [BL04, Proposition 4.1.2], the generic fiber $\mathcal{A}_\eta$ is isogenous to a principally polarized abelian variety A', defined over a finite extension of $\overline{\mathbb{Q}}(S)$. If we write this finite extension as $\overline{\mathbb{Q}}(S')$, with S' a smooth irreducible curve covering S, we can use Lemma 4.11 and assume that $S' = S$. By Proposition 7.3.6 and Theorem 7.4.5 of [BLR90], A' extends to an abelian scheme $\mathcal{A}' \to S$. Since S is smooth, using Lemma 4.12, we can then assume that the polarization induced by $\mathcal{L}$ is principal.

Then, by [Ge24, Lemma 2.2], there exists a finite étale cover $\ell : S' \to S$ such that $\mathcal{A}' := \mathcal{A} \times_S S' \to S'$ has level-3-structure.

Hence, since $\mathbb{A}_g = \mathbb{A}_{g,1,3}$ is a fine moduli space, there is a unique morphism $\varphi : S' \to \mathbb{A}_g$ such that $\mathcal{A}'$ is the pull-back of the universal family $\mathfrak{A}_g \to \mathbb{A}_g$ along φ. Thus, we have a cartesian diagram:

$$\begin{array}{ccccc} \mathcal{A}' & \xrightarrow{p'} & \mathfrak{A}_g & \xleftarrow{p''} & \mathcal{A}'' \\ \downarrow & & \downarrow & & \downarrow \\ S' & \xrightarrow{\varphi} & \mathbb{A}_g & \hookleftarrow & S'' \end{array}$$

Let $S'' = \varphi(S') \subseteq \mathbb{A}_g$. Since S' is an irreducible curve, $\varphi : S' \to S''$ is either constant or finite. However, φ cannot be constant, as $\mathcal{A} \to S$ would be isotrivial. Thus, φ is finite. Up to removing finitely many points from S', we can also assume that S'' is smooth, which implies that φ is flat.

Note that

$$\mathcal{A}' \cong \mathfrak{A}_g \times_{\mathbb{A}_g} S' \cong (\mathfrak{A}_g \times_{\mathbb{A}_g} S'') \times_{S''} S' = \mathcal{A}'' \times_{S''} S'$$

which gives a morphism $p : \mathcal{A}' \to \mathcal{A}''$.

Lemma 4.13. *Let $\mathcal{A}'' \to S''$ as above and $\mathcal{C}' \subseteq \mathcal{A}'$ be a curve satisfying the hypotheses of Theorem 4.1. Then, if the claim of Theorem 4.1 holds for $\mathcal{C}'' = p(\mathcal{C}')$, then it holds for $\mathcal{C}'$.*

Proof. We start by proving that the hypotheses of Theorem 4.1 hold for $\mathcal{C}''$. Firstly, $\mathcal{C}''$ cannot be contained in a fixed fiber $\mathcal{A}''_{s''}$, otherwise

$$\mathcal{C}' \subseteq p^{-1}(\mathcal{C}'') \subseteq \mathfrak{A}_{g,s''} \times \{s' \in S' : \varphi(s') = s''\}.$$

Since $\mathcal{C}'$ is irreducible and φ is finite, $\mathcal{C}' \subseteq \mathfrak{A}_{g,s''} \times \{s'\} = \mathcal{A}'_{s'}$, for some $s' \in S'$ such that $\varphi(s') = s''$, contradicting the assumptions on $\mathcal{C}'$. Furthermore, since φ is flat and finite, p is flat and finite as well. So, preimages by p of flat subgroup schemes of $\mathcal{A}''$ are flat subgroup schemes of $\mathcal{A}'$ of the same dimension, as in the proof of Lemma 4.11. This proves that $\mathcal{C}''$ is not contained in a proper flat subgroup scheme of $\mathcal{A}''$.

For a fiber $\mathcal{A}'_{s'} \cong \mathfrak{A}_{g,\varphi(s')}$, we have that $p(\mathcal{A}'_{s'}) = \mathcal{A}''_{\varphi(s')} = \mathfrak{A}_{g,\varphi(s')} \cong \mathcal{A}'_{s'}$. Also, images by p of subgroups of $\mathcal{A}'_{s'}$ are subgroups of $\mathcal{A}''_{\varphi(s')}$ of the same dimension. Therefore, the images of the intersections of $\mathcal{C}'$ with the union of the proper algebraic subgroups of the CM fibers are contained in the intersection of $\mathcal{C}''$ with the union of the proper algebraic subgroups of the CM fibers of $\mathcal{A}''$, which is a finite set by assumption. The conclusion follows by using the fact that p is finite. □

Thus, for the remainder of the chapter, we assume that $S \subseteq \mathbb{A}_g$ is a smooth, irreducible, locally closed curve defined over $\overline{\mathbb{Q}}$, and $\mathcal{A} = \mathfrak{A}_g \times_{\mathbb{A}_g} S$.

4.4 Matrix bounds for endomorphisms of abelian varieties

Let A be an abelian variety of dimension g defined over $\mathbb{C}$, so that $A \cong \mathbb{C}^g/\Lambda$ for some lattice Λ. Fix a polarization $\mathcal{L}$ of type $\mathbf{D} = \operatorname{diag}(d_1, \dots, d_g)$ and let $d = d_1 \cdot \ldots \cdot d_g$ be its degree. Fix also a symplectic basis $\lambda_1, \dots, \lambda_{2g}$ of Λ and a basis $e_1, \dots, e_g$ of $\mathbb{C}^g$ such that the period matrix of A with respect to these bases is $(\tau, \mathbf{D})$, where $\tau \in \mathbb{H}_g$ (see [BL04, Section 8.1]).

As in Section 4.2.1, denote by $\mathfrak{F}_g$ the fundamental domain for the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathbb{H}_g$, as described in [Igu72, Section V.4]. Fix a finite index subgroup Γ of $\mathrm{Sp}_{2g}(\mathbb{Z})$ and denote by $\mathfrak{F}_\Gamma$ the Siegel fundamental domain for Γ. Recall that $\mathfrak{F}_\Gamma$ was defined in (4.1) as $\mathfrak{F}_\Gamma = \bigcup_{i=1}^{n} \sigma_i \cdot \mathfrak{F}_g$, where $\sigma_1 = \mathbf{1}_{2g}, \sigma_2, \dots, \sigma_n \in \mathrm{Sp}_{2g}(\mathbb{Z})$ is a complete set of representatives for the right cosets of Γ in $\mathrm{Sp}_{2g}(\mathbb{Z})$.

In order to state and prove the result of this section, we introduce some matrix norms.

Definition 4.14. For a matrix $M = (m_{i,j})_{1 \le i,j \le n} \in \mathrm{Mat}_n(\mathbb{C})$ we define the following norms:

- $\|M\|_\infty := \max_{i,j} |m_{i,j}|$;
- (Frobenius norm) $\|M\|_F := \sqrt{\operatorname{tr}\left(\overline{M}^t M\right)} = \sqrt{\sum_{i,j=1}^{n} |m_{i,j}|^2}$;
- (Spectral norm) $\|M\|_2 := \sqrt{\rho\left(\overline{M}^t M\right)}$, where $\rho(M)$ denotes the spectral radius of M, i.e. the maximum of the absolute values of the eigenvalues of M.

Recall that the polarization $\mathcal{L}$ defines a Rosati involution † (see Equation (2.3)). Throughout, rational representations are taken with respect to the symplectic basis fixed above.

As established in [BL04, Theorem 5.1.8],

$$\operatorname{tr}(\rho_r(f^\dagger f)) > 0$$

for any nonzero $f \in \operatorname{End}^0(A) := \operatorname{End}(A) \otimes \mathbb{Q}$. Hence, $\|\rho_r(f)\|_\infty$ and $\sqrt{\operatorname{tr}\left(\rho_r(f^\dagger f)\right)}$ are two equivalent norms on the finite dimensional $\mathbb{Q}$-vector space $\operatorname{End}^0(A)$. Thus, there exist two constants $c_1, c_2 > 0$ such that

$$c_1 \cdot \sqrt{\operatorname{tr}\left(\rho_r(f^\dagger f)\right)} \leq \|\rho_r(f)\|_\infty \leq c_2 \cdot \sqrt{\operatorname{tr}\left(\rho_r(f^\dagger f)\right)}$$

for every $f \in \operatorname{End}^0(A)$. The aim of this section is to make the constants c_1, c_2 effective by proving the following result.

Proposition 4.15. *Let A be an abelian variety of dimension g defined over $\mathbb{C}$. Fix a polarization $\mathcal{L}$ and choose bases of Λ and $\mathbb{C}^g$ as above. Consider the Rosati involution † on $\operatorname{End}^0(A)$ defined by $\mathcal{L}$ and assume that $\tau \in \mathfrak{F}_\Gamma$. Then, for every $f \in \operatorname{End}^0(A)$, we have*

$$\frac{1}{2g \cdot c(A)} \cdot \sqrt{\operatorname{tr}\left(\rho_r(f^\dagger f)\right)} \leq \|\rho_r(f)\|_\infty \leq c(A) \cdot \sqrt{\operatorname{tr}\left(\rho_r(f^\dagger f)\right)}$$

where $c(A) = \delta(g, \mathfrak{F}_\Gamma) \cdot \dfrac{\|\mathbf{D}\|_\infty^{2g+2}}{d} \cdot \max\left\{1, \|\operatorname{Im}(Z_\tau)\|_\infty\right\}^{2g^3+3g^2+2g+1}$, $\delta(g, \mathfrak{F}_\Gamma)$ is an effective positive constant depending only on g and the choice of the representatives of the right cosets of Γ in $\operatorname{Sp}_{2g}(\mathbb{Z})$ and $Z_\tau \in \mathfrak{F}_g$ is in the same $\operatorname{Sp}_{2g}(\mathbb{Z})$-orbit as τ.

Let H be the Hermitian form associated with the polarization $\mathcal{L}$, and let $E = \operatorname{Im}(H)$ be the associated alternating form, which satisfies $E(\Lambda \times \Lambda) \subseteq \mathbb{Z}$. According to [BL04, Lemma 2.1.7], the form H can be expressed as:

$$H(u, v) = E(iu, v) + iE(u, v)$$

for every $u, v \in \mathbb{C}^g$, with $S(u, v) = E(iu, v) = \operatorname{Re}(H(u, v))$ positive definite. Let † be the Rosati involution defined by the polarization $\mathcal{L}$. By Proposition 5.1.1 of [BL04], we have

$$H(\rho_a(f)(u), v) = H(u, \rho_a(f^\dagger)(v))$$

for any $f \in \operatorname{End}^0(A)$ and for all $u, v \in \mathbb{C}^g$. As in [MW94], evaluating this expression at $\lambda_1, \ldots, \lambda_{2g}$ and taking real and imaginary parts yields

$$\rho_r(f^\dagger) = S^{-1} \cdot \rho_r(f)^t \cdot S = E^{-1} \cdot \rho_r(f)^t \cdot E$$

where, with a slight abuse of notation, we denote by S and E the matrices representing the bilinear forms $S(u, v)$ and $E(u, v)$ with respect to the basis $\lambda_1, \ldots, \lambda_{2g}$ of Λ. If we denote $R = \rho_r(f)$, then

$$\operatorname{tr}\left(\rho_r(f^\dagger f)\right) = \operatorname{tr}\left(S^{-1}R^t\, SR\right).$$

Since S is positive definite, there is a unique upper triangular matrix $U \in \operatorname{Mat}_{2g}(\mathbb{R})$ with positive diagonal entries such that $S = U^t U$. Substituting this decomposition, we have:

$$\begin{aligned} S^{-1}R^t\, SR &= U^{-1}(U^{-1})^t\, R^t\, U^t U R \\ &= U^{-1} \cdot \left((U^{-1})^t\, R^t\, U^t\right) \cdot \left(U R\, U^{-1}\right) \cdot U \end{aligned}$$

and the invariance of the trace under conjugation implies:

$$\operatorname{tr}\left(\rho_r(f^\dagger f)\right) = \operatorname{tr}\left(Q^t\, Q\right) = \|Q\|_F^2$$

where $Q = U R\, U^{-1}$.

Furthermore, by the triangle inequality, for any $M_1, M_2 \in \operatorname{Mat}_n(\mathbb{C})$ we have

$$\|M_1 M_2\|_\infty \leq n\, \|M_1\|_\infty \cdot \|M_2\|_\infty\,. \tag{4.3}$$

Therefore, since $R = U^{-1}QU$, we get

$$\|R\|_\infty = \left\|U^{-1}QU\right\|_\infty \leq \left((2g)^2 \cdot \left\|U^{-1}\right\|_\infty \cdot \|U\|_\infty\right) \cdot \|Q\|_\infty$$

$$\|Q\|_\infty = \left\|U R\, U^{-1}\right\|_\infty \leq \left((2g)^2 \cdot \left\|U^{-1}\right\|_\infty \cdot \|U\|_\infty\right) \cdot \|R\|_\infty\,.$$

We now prove a few general results about matrices.

Lemma 4.16. *If $M \in \operatorname{Mat}_n(\mathbb{R})$ is positive definite and $T \in \operatorname{Mat}_n(\mathbb{R})$ is an upper triangular matrix with positive diagonal entries such that $M = T^t \cdot T$, then $\|T\|_\infty \leq \sqrt{n\, \|M\|_\infty}$.*

Proof. We clearly have $\|M\|_2 = \|T\|_2^2$. Moreover, $\|N\|_\infty \leq \|N\|_2 \leq n\, \|N\|_\infty$ for every $N \in \operatorname{Mat}_n(\mathbb{R})$ [GVL13, Eq. (2.3.8)]. Thus, $\|T\|_\infty \leq \|T\|_2 = \sqrt{\|M\|_2} \leq \sqrt{n\, \|M\|_\infty}$. □

The following result is well known but we include it for completeness.

Lemma 4.17. *For any matrix $M \in \operatorname{Mat}_n(\mathbb{C})$, we have $|\det(M)| \leq n^{n/2} \cdot \|M\|_\infty^n$.*

Proof. This follows easily from Hadamard's inequality [Had93]. □

Lemma 4.18. *Let $M \in \operatorname{Mat}_n(\mathbb{C})$ be an invertible matrix. Then*

$$\left\|M^{-1}\right\|_\infty \leq \frac{n^{n/2}}{|\det(M)|} \cdot \|M\|_\infty^{n-1}\,.$$

Proof. The case $n = 1$ is trivial, so assume $n \geq 2$. Recall that $M^{-1} = \frac{1}{\det(M)} C^t$, where C is the cofactor matrix (see also the proof of Proposition 2.18 for details). By Lemma 4.17,

$\|C\|_\infty \leq (n-1)^{\frac{n-1}{2}} \cdot \|M\|_\infty^{n-1}$, which implies

$$\left\|M^{-1}\right\|_\infty = \frac{1}{|\det(M)|} \cdot \|C\|_\infty \leq \frac{(n-1)^{\frac{n-1}{2}}}{|\det(M)|} \cdot \|M\|_\infty^{n-1} \leq \frac{n^{n/2}}{|\det(M)|} \cdot \|M\|_\infty^{n-1}.$$

□

Next, we compute $S(u,v)$. Write $\tau = X + iY$, with $X = (x_{j,k})_{1\leq j,k\leq g}$ and $Y = (y_{j,k})_{1\leq j,k\leq g}$ real matrices. Recall that for the bases $(\lambda_1, \dots, \lambda_{2g})$ and $(e_1, \dots, e_g)$ that we fixed at the start we have

$$\lambda_j = \begin{cases} \sum_{k=1}^{g} x_{j,k} \cdot e_k + y_{j,k} \cdot ie_k & j = 1, \dots g \\ d_{j-g} e_{j-g} & j = g+1, \dots, 2g \end{cases}.$$

So, by doing the computations with the basis $(e_1, \dots, e_g, ie_1, \dots, ie_g)$ of $W = \Lambda \otimes \mathbb{R}$, the multiplication by i on W is represented in the basis $(\lambda_1, \dots, \lambda_{2g})$ by the matrix

$$\begin{pmatrix} X & \mathbf{D} \\ Y & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix} \begin{pmatrix} X & \mathbf{D} \\ Y & 0 \end{pmatrix} = \begin{pmatrix} Y^{-1}X & Y^{-1}\mathbf{D} \\ -\mathbf{D}^{-1}Y - \mathbf{D}^{-1}XY^{-1}X & -\mathbf{D}^{-1}XY^{-1}\mathbf{D} \end{pmatrix}.$$

Hence, the matrix representing $S(u,v) = E(iu,v)$ in the basis $(\lambda_1, \dots, \lambda_{2g})$ is given by

$$\begin{aligned} S &= \begin{pmatrix} Y^{-1}X & Y^{-1}\mathbf{D} \\ -\mathbf{D}^{-1}Y - \mathbf{D}^{-1}XY^{-1}X & -\mathbf{D}^{-1}XY^{-1}\mathbf{D} \end{pmatrix}^t \begin{pmatrix} 0 & \mathbf{D} \\ -\mathbf{D} & 0 \end{pmatrix} \\ &= \begin{pmatrix} XY^{-1}X + Y & XY^{-1}\mathbf{D} \\ \mathbf{D}Y^{-1}X & \mathbf{D}Y^{-1}\mathbf{D} \end{pmatrix}. \end{aligned}$$

Furthermore, note that by [AM05, Ex. 5.30]

$$\begin{aligned} \det(S) &= \det(\mathbf{D}Y^{-1}\mathbf{D}) \det\Big((XY^{-1}X + Y) - (XY^{-1}\mathbf{D})(\mathbf{D}Y^{-1}\mathbf{D})^{-1}(\mathbf{D}Y^{-1}X)\Big) \\ &= \det(\mathbf{D})^2 \cdot \det(Y^{-1}) \cdot \det(Y) = \det(\mathbf{D})^2 = d^2 \end{aligned}$$

which also implies that $\det(U) = d$, since $S = U^tU$ and U has positive diagonal entries.

Then, by Lemma 4.16, we have that $\|U\|_\infty \leq \sqrt{2g\,\|S\|_\infty}$ and using Lemma 4.18 we get

$$\begin{aligned} \left\|U^{-1}\right\|_\infty &\leq \frac{(2g)^g}{d} \cdot \|U\|_\infty^{2g-1} \\ &\leq \frac{(2g)^g}{d} \cdot (2g \max\{1, \|S\|_\infty\})^g = \frac{(2g)^{2g}}{d} \cdot \max\{1, \|S\|_\infty\}^g. \end{aligned} \tag{4.4}$$

Finally, in preparation for the proof of Proposition 4.15, we establish some bounds for matrices in $\mathfrak{F}_\Gamma$. To this end, we first recall a few classical properties of the Siegel fundamental domain $\mathfrak{F}_g$.

Lemma 4.19. *Let $\tau = X + iY \in \mathfrak{F}_g$. Then, we have:*

(a) $\|X\|_\infty \leq \frac{1}{2}$;

(b) $\det(Y) \geq \left(\frac{\sqrt{3}}{2}\right)^{g^2}$;

(c) $|\det(C\tau + D)| \geq 1$, *for every* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{Z})$.

Proof. Parts (a) and (c) are true by definition of $\mathfrak{F}_g$ (see [Igu72, p. 194]). Moreover, by Lemmas V.13 and V.15 of [Igu72]

$$\det(Y) \geq \left(\frac{3}{4}\right)^{\frac{g(g-1)}{2}} \cdot (y_{1,1})^g \geq \left(\frac{3}{4}\right)^{\frac{g(g-1)}{2}} \cdot \left(\frac{\sqrt{3}}{2}\right)^g = \left(\frac{\sqrt{3}}{2}\right)^{g^2}$$

which proves part (b). □

Proposition 4.20. *Let $\tau = X + iY \in \mathfrak{F}_\Gamma$ and let $Z_\tau \in \mathfrak{F}_g$ be in the same $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit as τ. Then, there are effective positive constants $\delta_1, \delta_2, \delta_3, \delta_4$, depending only on g and the choices of the representatives for the right cosets of Γ in $\mathrm{Sp}_{2g}(\mathbb{Z})$, such that:*

(a) $\|Y\|_\infty \leq \delta_1 \cdot \max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\}^{2g-1}$;

(b) $\|X\|_\infty \leq \delta_2 \cdot \max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\}^{g}$;

(c) $\det(Y) \geq \dfrac{\delta_3}{\max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\}^{2g}}$;

(d) $\|Y^{-1}\|_\infty \leq \delta_4 \cdot \max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\}^{2g^2-g+1}$.

Proof. Let τ and Z_τ as above and take $\sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $\tau = \sigma \cdot Z_\tau$. The definition of $\mathfrak{F}_\Gamma$ (see (4.1)) implies that we can take σ to be one of the chosen representatives $\sigma_1, \ldots, \sigma_n$ for the right cosets of Γ in $\mathrm{Sp}_{2g}(\mathbb{Z})$ and thus all the constants that appear will depend on the choice of such representatives.

(a) It is well-known that

$$Y = \mathrm{Im}(\tau) = \mathrm{Im}(\sigma \cdot Z_\tau) = \left[(CZ_\tau + D)^t\right]^{-1} \mathrm{Im}(Z_\tau) \left(C\overline{Z_\tau} + D\right)^{-1} \tag{4.5}$$

(see for example [Igu72, Section I.6]). So (4.3) implies that

$$\|Y\|_\infty \leq g^2 \cdot \left\|(CZ_\tau + D)^{-1}\right\|_\infty^2 \cdot \|\mathrm{Im}(Z_\tau)\|_\infty$$

since $\|\cdot\|_\infty$ is invariant under transposition and complex conjugation. Then, as $Z_\tau \in \mathfrak{F}_g$, Lemma 4.18 and Lemma 4.19(c) imply

$$\left\|(CZ_\tau + D)^{-1}\right\|_\infty \leq \frac{g^{g/2}}{|\det(CZ_\tau + D)|} \cdot \|CZ_\tau + D\|_\infty^{g-1} \leq g^{g/2} \cdot \|CZ_\tau + D\|_\infty^{g-1}.$$

Moreover,

$$\begin{aligned} \|CZ_\tau + D\|_\infty &\le g\,\|C\|_\infty\,\|Z_\tau\|_\infty + \|D\|_\infty \\ &\le \frac{5g}{2}\cdot \max\{\|C\|_\infty, \|D\|_\infty\}\cdot \max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\} \end{aligned} \tag{4.6}$$

since Lemma 4.19(a) implies

$$\begin{aligned} \|Z_\tau\|_\infty + 1 &\le \|\mathrm{Re}(Z_\tau)\|_\infty + \|\mathrm{Im}(Z_\tau)\|_\infty + 1 \\ &\le \frac{3}{2} + \|\mathrm{Im}(Z_\tau)\|_\infty \le \frac{5}{2}\cdot \max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\}. \end{aligned} \tag{4.7}$$

Combining the inequalities above yields

$$\begin{aligned} \|Y\|_\infty &\le g^2\cdot \left\|(CZ_\tau + D)^{-1}\right\|_\infty^2 \cdot \|\mathrm{Im}(Z_\tau)\|_\infty \\ &\le g^{g+2}\cdot \|CZ_\tau + D\|_\infty^{2g-2}\cdot \|\mathrm{Im}(Z_\tau)\|_\infty \\ &\le \left(\frac{5}{2}\right)^{2g-2}\cdot g^{3g}\cdot \max\{\|C\|_\infty, \|D\|_\infty\}^{2g-2}\cdot \max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\}^{2g-1}. \end{aligned}$$

Hence we can take $\delta_1 = \left(\frac{5}{2}\right)^{2g-2}\cdot g^{3g}\cdot \max_{\sigma\in\{\sigma_1,\ldots,\sigma_n\}}\{\max\{\|C\|_\infty, \|D\|_\infty\}\}^{2g-2}$.

(b) We have that

$$\begin{aligned} \|X\|_\infty = \|\mathrm{Re}(\sigma\cdot Z_\tau)\|_\infty \le \|\sigma\cdot Z_\tau\|_\infty &= \left\|(AZ_\tau + B)(CZ_\tau + D)^{-1}\right\|_\infty \\ &\le g\cdot \|AZ_\tau + B\|_\infty \cdot \left\|(CZ_\tau + D)^{-1}\right\|_\infty \\ &\le g\cdot (g\,\|A\|_\infty\,\|Z_\tau\|_\infty + \|B\|_\infty)\cdot \left\|(CZ_\tau + D)^{-1}\right\|_\infty. \end{aligned}$$

From the computations above we also have that

$$\begin{aligned} \left\|(CZ_\tau + D)^{-1}\right\|_\infty &\le g^{g/2}\cdot \|CZ_\tau + D\|_\infty^{g-1} \\ &\le \left(\frac{5}{2}\right)^{g-1} g^{\frac{3}{2}g-1}\max\{\|C\|_\infty, \|D\|_\infty\}^{g-1}\max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\}^{g-1}. \end{aligned}$$

This implies that

$$\|X\|_\infty \le \left(\frac{5}{2}\right)^{g} g^{\frac{3}{2}g+1}\,\|\sigma\|_\infty^g \max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\}^g.$$

Hence, we can take $\delta_2 = \left(\frac{5}{2}\right)^{g} g^{\frac{3}{2}g+1}\max_{\sigma\in\{\sigma_1,\ldots,\sigma_n\}}\{\|\sigma\|_\infty\}^g$.

(c) Taking the determinant of the first and last part of Equation (4.5) yields

$$\det(Y) = \det(CZ_\tau + D)^{-1}\cdot \det(\mathrm{Im}(Z_\tau))\cdot \det(C\overline{Z_\tau} + D)^{-1} = \frac{\det(\mathrm{Im}(Z_\tau))}{|\det(CZ_\tau + D)|^2}.$$

Furthermore, it follows from Lemma 4.17 and Equation (4.6) that

$$
\begin{aligned}
|\det(CZ_\tau + D)| &\le g^{g/2} \cdot \|CZ_\tau + D\|_\infty^g \\
&\le \left(\frac{5}{2}\right)^g \cdot g^{3g/2} \cdot \max\{\|C\|_\infty, \|D\|_\infty\}^g \cdot \max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\}^g .
\end{aligned}
$$

Therefore, using Lemma 4.19(b), we get

$$
\det(Y) = \frac{\det(\mathrm{Im}(Z_\tau))}{|\det(CZ_\tau + D)|^2} \ge \frac{\frac{\left(\frac{\sqrt{3}}{2}\right)^{g^2}}{\left(\frac{5}{2}\right)^{2g} \cdot g^{3g} \cdot \max\left\{\|C\|_\infty, \|D\|_\infty\right\}^{2g}}}{\max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\}^{2g}}
$$

so that we can take $\delta_3 = \left(\frac{\sqrt{3}}{2}\right)^{g^2} \left(\frac{2}{5}\right)^{2g} \cdot g^{-3g} \cdot \max_{\sigma \in \{\sigma_1, \ldots, \sigma_n\}} \{\max\{\|C\|_\infty, \|D\|_\infty\}\}^{-2g}$.

(d) Applying Lemma 4.18 and parts (a) and (c) yields

$$
\left\|Y^{-1}\right\|_\infty \le \frac{g^{g/2}}{|\det(Y)|} \cdot \|Y\|_\infty^{g-1} \le g^{g/2} \cdot \frac{\gamma_1^{g-1}}{\gamma_3} \cdot \max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\}^{2g^2 - g + 1} .
$$

Thus, we can take $\delta_4 = g^{g/2} \cdot \frac{\delta_1^{g-1}}{\delta_3}$.

□

We are now ready to prove the main result of this section.

Proof of Proposition 4.15. We already proved that

$$
\|R\|_\infty \le \left((2g)^2 \cdot \left\|U^{-1}\right\|_\infty \cdot \|U\|_\infty\right) \cdot \|Q\|_\infty , \qquad \|Q\|_\infty \le \left((2g)^2 \cdot \left\|U^{-1}\right\|_\infty \cdot \|U\|_\infty\right) \cdot \|R\|_\infty .
$$

Now, by Equation (4.4), we have that

$$
\begin{aligned}
(2g)^2 \cdot \left\|U^{-1}\right\|_\infty \cdot \|U\|_\infty &\le (2g)^2 \cdot \frac{(2g)^{2g}}{d} \cdot \max\{1, \|S\|_\infty\}^g \cdot (2g)^{1/2} \cdot \|S\|_\infty^{1/2} \\
&\le \frac{(2g)^{2g+3}}{d} \cdot \max\{1, \|S\|_\infty\}^{g+1} .
\end{aligned}
$$

Let $\tau = X + iY$. Then, using (4.3), we get

$$
\begin{aligned}
\|S\|_\infty &= \max\left\{\left\|XY^{-1}X + Y\right\|_\infty, \left\|XY^{-1}\mathbf{D}\right\|_\infty, \left\|\mathbf{D}Y^{-1}X\right\|_\infty, \left\|\mathbf{D}Y^{-1}\mathbf{D}\right\|_\infty\right\} \\
&\le \max\left\{\|Y\|_\infty + g^2 \|X\|_\infty^2 \left\|Y^{-1}\right\|_\infty, g^2 \|X\|_\infty \left\|Y^{-1}\right\|_\infty \|\mathbf{D}\|_\infty, g^2 \|\mathbf{D}\|_\infty^2 \left\|Y^{-1}\right\|_\infty\right\} .
\end{aligned}
$$

Moreover, if $\tau \in \mathfrak{F}_\Gamma$, let $Z_\tau \in \mathfrak{F}_g$ be in the same $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit as τ, as before. Then, by Proposition 4.20, we also obtain:

$$
\|Y\|_\infty + g^2 \|X\|_\infty^2 \left\|Y^{-1}\right\|_\infty \le 2g^2 \delta_2^2 \delta_4 \cdot \max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\}^{2g^2 + g + 1} ,
$$

$$g^2 \|X\|_\infty \left\|Y^{-1}\right\|_\infty \|\mathbf{D}\|_\infty \leq g^2\delta_2\delta_4 \|\mathbf{D}\|_\infty \cdot \max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\}^{2g^2+1},$$
$$g^2 \|\mathbf{D}\|_\infty^2 \left\|Y^{-1}\right\|_\infty \leq g^2\delta_4 \|\mathbf{D}\|_\infty^2 \cdot \max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\}^{2g^2-g+1}.$$

So, we get

$$\begin{aligned}\|S\|_\infty &\leq \max\left\{\|Y\|_\infty + g^2 \|X\|_\infty^2 \left\|Y^{-1}\right\|_\infty, g^2 \|X\|_\infty \left\|Y^{-1}\right\|_\infty \|\mathbf{D}\|_\infty, g^2 \|\mathbf{D}\|_\infty^2 \left\|Y^{-1}\right\|_\infty\right\} \\ &\leq 2g^2\delta_2^2\delta_4 \|\mathbf{D}\|_\infty^2 \cdot \max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\}^{2g^2+g+1}.\end{aligned}$$

Thus,

$$(2g)^2 \cdot \left\|U^{-1}\right\|_\infty \cdot \|U\|_\infty \leq \frac{(2g)^{2g+3}}{d} \cdot \max\{1, \|S\|_\infty\}^{g+1} \leq c(A)$$

where

$$c(A) = 2^{2g+4} \cdot g^{2g+5} \cdot \delta_2^2\delta_4 \cdot \frac{\|\mathbf{D}\|_\infty^{2g+2}}{d} \cdot \max\{1, \|\mathrm{Im}(Z_\tau)\|_\infty\}^{2g^3+3g^2+2g+1}.$$

Note that $\delta = 2^{2g+4} \cdot g^{2g+5} \cdot \delta_2^2\delta_4$ is an effective positive constant that depends only on g and the choice of the representatives for the right cosets of Γ in $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Therefore, we have that

$$\|R\|_\infty \leq c(A) \cdot \|Q\|_\infty \leq c(A) \cdot \|Q\|_F$$
$$\frac{1}{2g} \|Q\|_F \leq \|Q\|_\infty \leq c(A) \cdot \|R\|_\infty.$$

Recalling that $R = \rho_r(f)$ and $\|Q\|_F = \sqrt{\mathrm{tr}\left(\rho_r(f^\dagger f)\right)}$ concludes the proof. □

Remark 4.21. If $\Gamma = \mathrm{Sp}_{2g}(\mathbb{Z})$ (so that $\mathfrak{F}_\Gamma = \mathfrak{F}_g$ and $Z_\tau = \tau$), one can obtain a better value for the constant $c(A)$, namely

$$c(A) = 2^{4g+5} \cdot g^{g^2+3g+3} \cdot \left(\frac{2\sqrt{3}}{3}\right)^{g^2(g+1)} \cdot \frac{\|\mathbf{D}\|_\infty^{2g+2}}{d} \cdot \max\{1, \|\mathrm{Im}(\tau)\|_\infty\}^{g(g+1)}.$$

The argument is the same as in the proof above, but here one may use the sharper bounds specific to $\mathfrak{F}_g$ given by Lemma 4.19 instead of Proposition 4.20.

4.5 The main estimate

For every $T \geq 1$ we define the set

$$\begin{aligned}\mathcal{Z}(T) = \Big\{ &(\tau, z) \in \mathcal{Z} : \exists M \in \mathrm{Mat}_g(\mathbb{C}) \setminus \{\mathbf{0}\} \text{ s.t.} \\ &Mz \in \mathbb{Z}^g + \tau\mathbb{Z}^g, H_{2g}(\tau), H_{2g}(M) \leq T \text{ and } \det(\mathrm{Im}(\tau)) \geq \frac{1}{T}\Big\}\end{aligned}$$

where $\mathcal{Z}$ is the set defined in (4.2) and H_{2g} is the height defined in Definition 4.6.

We want to prove the following upper bound for the cardinality of $\mathcal{Z}(T)$.

Proposition 4.22. *Under the hypotheses of Theorem 4.1, for all $\varepsilon > 0$, we have $\#\mathcal{Z}(T) \ll_\varepsilon T^\varepsilon$, for all $T \geq 1$.*

In order to prove this, consider the definable set W whose elements are tuples of the form

$$(\alpha_{1,1},\ldots,\alpha_{g,g},\beta_{1,1},\ldots,\beta_{g,g},\mu_{1,1},\ldots,\mu_{1,g},\mu_{2,1},\ldots,\mu_{2,g},\\ x_{1,1},\ldots,x_{g,g},y_{1,1},\ldots,y_{g,g},u_1,\ldots,u_g,v_1,\ldots,v_g)$$

in $\mathbb{R}^{4g^2+2g}\times\mathbb{R}^{2g}$, satisfying the following relations:

$$M \neq \mathbf{0}, \qquad (\tau, z) \in \mathcal{Z}, \qquad Mz = \mu_1 + \tau\mu_2$$

where

$$M = (\alpha_{i,j} + \mathbf{i}\beta_{i,j})_{i,j=1,\ldots,g}\,, \quad \mu_1 = (\mu_{1,1},\ldots,\mu_{1,g})^t\,, \quad \mu_2 = (\mu_{2,1},\ldots,\mu_{2,g})^t\,,$$

$$\tau = (x_{i,j} + \mathbf{i}y_{i,j})_{i,j=1,\ldots,g}\,, \quad z = (z_1,\ldots,z_g)^t = (u_1 + \mathbf{i}v_1,\ldots,u_g+\mathbf{i}v_g)^t$$

and $\mathbf{i}$ is the imaginary unit. In particular, for $T \geq 1$, let

$$W^\sim(2g,T) := \{(\alpha_{1,1},\ldots,v_g) \in W : H_{2g}(\alpha_{1,1},\ldots,y_{g,g}) \leq T\}\,.$$

Recall that $H_{2g}(\alpha_{1,1},\ldots,y_{g,g})$ is finite if and only if $\alpha_{1,1},\ldots,y_{g,g}$ are all algebraic numbers of degree at most $2g$.

Now, let π_1,π_2 be the projections on the first $4g^2+2g$ and the last $2g$ coordinates, respectively.

Lemma 4.23. *Under the hypotheses of Theorem 4.1, for every $\varepsilon > 0$, we have*

$$\#\pi_2\left(W^\sim(2g,T)\right) \ll_\varepsilon T^\varepsilon$$

for all $T \geq 1$.

Proof. Consider an arbitrary $\varepsilon > 0$ and assume that for some $T_0 \geq 1$, $\#\pi_2\left(W^\sim(2g,T_0)\right) > cT_0^\varepsilon$, where $c = c(W, 2g, \varepsilon)$ is the constant given by Proposition 4.10.

Then, by Proposition 4.10, there exists a continuous definable function $\delta : [0,1] \to W$ such that $\delta_1 = \pi_1 \circ \delta : [0,1] \to \mathbb{R}^{4g^2+2g}$ is semi-algebraic and $\delta_2 = \pi_2\circ\delta : [0,1] \to \mathbb{R}^{2g}$ is non-constant. Hence, there exists an infinite connected $J \subseteq [0,1]$ such that $\delta_1(J)$ is contained in an algebraic curve and $\delta_2(J)$ has positive dimension.

Let $M, \tau, \mu_1, \mu_2, z = (z_1, \ldots, z_g)^t$ be as above and consider the coordinates

$$\alpha_{1,1}, \ldots, \alpha_{g,g}, \beta_{1,1}, \ldots, \beta_{g,g}, \mu_{1,1}, \ldots, \mu_{1,g}, \mu_{2,1}, \ldots, \mu_{2,g},$$
$$x_{1,1}, \ldots, x_{g,g}, y_{1,1}, \ldots, y_{g,g}, u_1, \ldots, u_g, v_1, \ldots, v_g$$

as functions on J.

Note that τ cannot be constant on J, otherwise there would be infinitely many points on $\mathcal{C}$ (since $\delta_2(J)$ has positive dimension) that lie on the same fiber, which contradicts the assumption that $\mathcal{C}$ is not contained in any fiber.

Moreover, on J, the functions $\alpha_{1,1}, \ldots, y_{g,g}$ generate a field of transcendence degree at most 1 over $\mathbb{C}$, because they are functions on a curve. Therefore, on J, $\mathbb{C}(\tau)$ is a field of transcendence degree 1 over $\mathbb{C}$ and $\alpha_{1,1}, \ldots, \mu_{2,g} \in \overline{\mathbb{C}(\tau)}$. Since $M \neq \mathbf{0}$ and $Mz = \mu_1 + \tau\mu_2$, it follows that $z_1, \ldots, z_g$ are linearly dependent over $\overline{\mathbb{C}(\tau)}$. In particular, $z_1, \ldots, z_g$ are algebraically dependent over $F = \mathbb{C}(\tau)$ on J.

Now, consider the set $\mathcal{W} = (\tau, z)(J) \subseteq \mathcal{Z}$. As the restriction of δ to $(0, 1)$ is real analytic, we can view τ and z as holomorphic functions on $u(\mathcal{W}) \subseteq \mathcal{C}(\mathbb{C})$. Then, τ and z satisfy an algebraic relation on $u(\mathcal{W})$ which can be analytically continued to an open disc in $\mathcal{C}(\mathbb{C})$.

Therefore, we have $\operatorname{tr.deg.}_F F(z) < g$ on an open disc in $\mathcal{C}(\mathbb{C})$, contradicting Lemma 4.5 and thus proving the proposition. □

Lemma 4.24. *There exists a positive constant $c' = c'(\mathcal{Z})$ such that for all $z \in \mathbb{C}^g$ and for all $T \geq 1$, there are at most c' elements $\tau \in \mathbb{H}_g$ such that $(\tau, z) \in \mathcal{Z}(T)$.*

Proof. Let

$$\begin{aligned} \widetilde{\pi}: \ \mathcal{Z} &\longrightarrow \mathbb{C}^g \\ (\tau, z) &\longmapsto z \end{aligned}$$

Fix $z_0 \in \mathbb{C}^g$. By o-minimality, if $\widetilde{\pi}^{-1}(z_0)$ has dimension 0, then Proposition 3.10 implies that its cardinality is uniformly bounded by a constant depending only on $\mathcal{Z}$. Therefore, it suffices to show that for any $T \geq 1$, if $z_0 \in \widetilde{\pi}(\mathcal{Z}(T))$, then $\widetilde{\pi}^{-1}(z_0)$ has dimension 0.

Now suppose that it has positive dimension, and let $\tau_0 \in \mathbb{H}_g$ be such that $(\tau_0, z_0) \in \mathcal{Z}(T)$. Then z_0 and τ_0 are algebraically dependent over $\mathbb{C}$, and this relation extends to the whole $\widetilde{\pi}(z_0)$, hence to an open disc in $\mathcal{C}(\mathbb{C})$. This contradicts Lemma 4.5. □

Proof of Proposition 4.22. If $(\tau, z) \in \mathcal{Z}(T)$, then there exists a matrix $M \in \operatorname{Mat}_g(\overline{\mathbb{Q}})$ satisfying $H_{2g}(M) \leq T$, and vectors $\mu_1, \mu_2 \in \mathbb{Z}^g$ such that

$$Mz = \mu_1 + \tau\mu_2.$$

If we write $M = (m_{i,j})_{1 \leq i,j \leq g}$ and $\tau = (\tau_{i,j})_{1 \leq i,j \leq g}$, then, for every $i, j = 1, \ldots, g$, we can use Lemma 4.7 and deduce

$$|m_{i,j}| \leq \sqrt{2g+1}H_{2g}(M) \ll T, \qquad |\tau_{i,j}| \leq \sqrt{2g+1}H_{2g}(\tau) \ll T.$$

Furthermore, since $z = (z_1, \dots, z_g) \in L_\tau$, there exist $u, v \in [0,1)^g$ such that $z = u + \tau v$. Thus, for each $i = 1, \dots, g$, we get

$$|z_i| = \left| u_i + \sum_{j=1}^{g} \tau_{i,j} v_j \right| \leq 1 + \sum_{j=1}^{g} |\tau_{i,j}| \ll T.$$

As a consequence, for every $i = 1, \dots, g$ we have

$$\left| \sum_{j=1}^{g} m_{i,j} z_j \right| \leq \sum_{j=1}^{g} |m_{i,j}| \, |z_j| \ll T^2. \tag{4.8}$$

Since $Mz = \mu_1 + \tau\mu_2$, we have $\mathrm{Im}(\tau)\mu_2 = \mathrm{Im}(Mz)$ and thus

$$\|\mu_2\|_\infty = \left\| \mathrm{Im}(\tau)^{-1} \cdot \mathrm{Im}(Mz) \right\|_\infty \leq g \left\| \mathrm{Im}(\tau)^{-1} \right\|_\infty \cdot \|\mathrm{Im}(Mz)\|_\infty$$

and, by Lemma 4.18, we get $\|\mathrm{Im}(\tau)^{-1}\|_\infty \leq \frac{g^{g/2}}{\det(\mathrm{Im}(\tau))} \|\mathrm{Im}(\tau)\|_\infty^{g-1} \leq g^{g/2} \cdot T \cdot \|\tau\|_\infty^{g-1} \ll T^g$. Hence, using (4.8), we obtain

$$\|\mu_2\|_\infty \leq g \left\| \mathrm{Im}(\tau)^{-1} \right\|_\infty \cdot \|\mathrm{Im}(Mz)\|_\infty \ll T^g \cdot \|Mz\|_\infty \ll T^{g+2}.$$

Moreover, we have $\mu_1 = Mz - \tau\mu_2$, so that

$$\|\mu_1\|_\infty \leq \|Mz\|_\infty + \|\tau\mu_2\|_\infty \leq \|Mz\|_\infty + g\,\|\tau\|_\infty \cdot \|\mu_2\|_\infty \ll T^2 + T \cdot T^{g+2} \ll T^{g+3}.$$

This allows us to deduce that

$$(\mathrm{Re}(M), \mathrm{Im}(M), \mu_1, \mu_2, \mathrm{Re}(\tau), \mathrm{Im}(\tau), \mathrm{Re}(z), \mathrm{Im}(z)) \in W^{\sim}(2g, \nu T^{g+3})$$

for some positive constant ν. Then, by Lemma 4.24, each element of $\pi_2(W^{\sim}(2g, \nu T^{g+3}))$ corresponds to at most c' distinct elements of $\mathcal{Z}(T)$. Finally, the proof follows from Lemma 4.23. □

4.6 A height inequality

The aim of this section is to give a bound on the canonical height of the points $P \in \mathcal{C}(\overline{\mathbb{Q}})$ in terms of the Faltings height $h_F(\mathcal{A}_{\pi(P)})$ of the corresponding fiber. In order to do that we recall the setting of Theorem 4.1 and the reductions made in Section 4.3, and also define some height functions that will be used to prove this bound.

Let $S \subseteq \mathbb{A}_g = \mathbb{A}_{g,\mathbf{1},3}$ be a smooth, irreducible, locally closed curve defined over $\overline{\mathbb{Q}}$, let $\mathcal{A} = \mathfrak{A}_g \times_{\mathbb{A}_g} S$, with $\pi : \mathcal{A} \to S$ being the structural morphism, and let $\mathcal{C} \subseteq \mathcal{A}$ be an irreducible curve as in Theorem 4.1. Recall that $\mathcal{A}$ has a level-3-structure and that there is a principal polarization $\lambda : \mathcal{A} \to \widehat{\mathcal{A}}$, where $\widehat{\mathcal{A}}$ denotes the dual abelian scheme of $\mathcal{A}$.

By [GW23, Proposition 27.284], the pullback of the Poincaré bundle $\mathscr{P}$ via the morphism $(\mathrm{id}_{\mathcal{A}}, \lambda)$ is relatively ample. Thus, the line bundle

$$\mathcal{L} = [(\mathrm{id}_{\mathcal{A}}, \lambda)^* \mathscr{P} \otimes [-1]^*_{\mathcal{A}} (\mathrm{id}_{\mathcal{A}}, \lambda)^* \mathscr{P}]^{\otimes 3}$$

is relatively very ample (see [GW23, Theorem 27.279]), symmetric and $\Phi_{\mathcal{L}} = 12\lambda$. This line bundle gives an embedding $\mathcal{A} \hookrightarrow \mathbb{P}^n_S \cong \mathbb{P}^n_{\overline{\mathbb{Q}}} \times S$. Moreover, for every fiber $\mathcal{A}_s$ of $\mathcal{A} \to S$, the induced closed immersion $\mathcal{A}_s \to \mathbb{P}^n_{\overline{\mathbb{Q}}}$ comes from the restriction $\mathcal{L}_s = \mathcal{L}|_{\mathcal{A}_s}$.

The minimal compactification $\overline{\mathbb{A}_{g,\mathbf{1},3}}$ of $\mathbb{A}_{g,\mathbf{1},3}$ can be realized as a closed subvariety of some projective space $\mathbb{P}^m_{\overline{\mathbb{Q}}}$ and we define $\mathcal{M} = \mathcal{O}_{\mathbb{P}^m}(1)|_{\overline{\mathbb{A}_{g,\mathbf{1},3}}}$. Thus, we obtain an embedding $\mathbb{A}_{g,\mathbf{1},3} \hookrightarrow \mathbb{P}^m_{\overline{\mathbb{Q}}}$ and we denote by $\overline{S}$ the Zariski closure of S in $\overline{\mathbb{A}_{g,\mathbf{1},3}} \subseteq \mathbb{P}^m_{\overline{\mathbb{Q}}}$.

We then denote by $\overline{\mathcal{A}}$ the Zariski closure of $\mathcal{A}$ inside $\mathbb{P}^n_{\overline{\mathbb{Q}}} \times \overline{S} \subseteq \mathbb{P}^n_{\overline{\mathbb{Q}}} \times \mathbb{P}^m_{\overline{\mathbb{Q}}}$ and let $\overline{\mathcal{L}} = \mathcal{O}(1,1)|_{\overline{\mathcal{A}}} = \mathcal{L} \otimes \pi^*(\mathcal{M}|_{\overline{S}})$. Using the properties of the Weil height (e.g. Theorem 2.22 or [HS13, Theorem B.3.6]), we define the *naive height* on $\mathcal{A}(\overline{\mathbb{Q}})$ as

$$h_{\overline{\mathcal{A}},\overline{\mathcal{L}}}(P) = h_{\mathcal{A}_{\pi(P)},\mathcal{L}_{\pi(P)}}(P) + h_{\overline{S},\mathcal{M}|_{\overline{S}}}(\pi(P)).$$

Moreover, as $\mathcal{L}$ is symmetric, we can also define a fiberwise canonical height $\widehat{h}_{\mathcal{A}_{\pi(P)},\mathcal{L}_{\pi(P)}}(P)$ as in Theorem 2.29.

Furthermore, recall that the coarse moduli space $\mathbb{A}_{g,\mathbf{1}}$ of principally polarized abelian varieties of dimension g is a quasi-projective variety. More precisely, its minimal compactification $\overline{\mathbb{A}_{g,\mathbf{1}}}$ can be realized as a closed subvariety of some projective space $\mathbb{P}^{\ell}_{\overline{\mathbb{Q}}}$.

Let $L = \mathcal{O}_{\mathbb{P}^{\ell}}(1)|_{\overline{\mathbb{A}_{g,\mathbf{1}}}}$. Then, by [FW12, Section II.3], L has an Hermitian metric on $\mathbb{A}_{g,\mathbf{1}}$ with logarithmic singularities along $\overline{\mathbb{A}_{g,\mathbf{1}}} \setminus \mathbb{A}_{g,\mathbf{1}}$. Hence, we can define two height functions: h_L on $\mathbb{A}_{g,\mathbf{1}}$ using the metric cited just now; and $\widetilde{h}_L$ on $\overline{\mathbb{A}_{g,\mathbf{1}}}$ given by the Hermitian metric which at the archimedean places is the standard Fubini–Study metric coming from the embedding of $\overline{\mathbb{A}_{g,\mathbf{1}}}$ into $\mathbb{P}^{\ell}_{\overline{\mathbb{Q}}}$ and at the non-archimedean places is the usual metric. Note that $\widetilde{h}_L$ differs from a fixed Weil height $h_{\overline{\mathbb{A}_{g,\mathbf{1}}},L}$ by a bounded function on $\mathbb{P}^{\ell}(\overline{\mathbb{Q}})$ (see [BG06, Remark 2.8.3]).

From this point forward, $\xi_1, \xi_2, \ldots$ will be positive constants depending only on g, S, $\mathcal{A}$, $\mathcal{C}$ and the choices of the various Weil heights, unless otherwise specified.

Proposition 4.25. *There exist positive constants ξ_1, ξ_2 such that*

$$\widehat{h}_{\mathcal{A}_{\pi(P)},\mathcal{L}_{\pi(P)}}(P) \leq \xi_1 h_F(\mathcal{A}_{\pi(P)}) + \xi_2$$

for every $P \in \mathcal{C}(\overline{\mathbb{Q}})$.

Proof. By [FW12, Theorem II.3.1] there exist positive constants ξ_3, ξ_4 depending only on g such that

$$|h_L([A]) - \xi_3 \cdot h_F(A)| \leq \xi_4$$

for every principally polarized $A/\overline{\mathbb{Q}}$ of dimension g. Here, we denote by $[A]$ the isomorphism class of A in $\mathbb{A}_{g,\mathbf{1}}$. By [FW12, Lemma II.1.2, last displayed equation], there are positive constants ξ_5, ξ_6, depending only on g, such that

$$\left|h_L([A]) - \widetilde{h}_L([A])\right| \leq \xi_5 + \xi_6 \log \max\left\{1, \widetilde{h}_L([A])\right\}$$

for each $[A] \in \mathbb{A}_{g,\mathbf{1}}$. In particular, this means that $\widetilde{h}_L([A]) \ll h_L([A]) + 1$, which combined with the inequality above yields $\widetilde{h}_L([A]) \ll h_F(A) + 1$. As noted above, $\widetilde{h}_L$ differs from $h_{\overline{\mathbb{A}_{g,\mathbf{1}}},L}$ by a bounded function, so we get

$$h_{\overline{\mathbb{A}_{g,\mathbf{1}}},L}([A]) \ll h_F(A) + 1 \tag{4.9}$$

for every principally polarized $A/\overline{\mathbb{Q}}$ of dimension g, where the implied constant depends only on g and the choice of the Weil height $h_{\overline{\mathbb{A}_{g,\mathbf{1}}},L}$.

Let $\rho : \mathbb{A}_{g,\mathbf{1},3} \to \mathbb{A}_{g,\mathbf{1}}$ be the natural morphism which forgets the level structure. It extends to a rational map

$$\overline{\rho} : \overline{\mathbb{A}_{g,\mathbf{1},3}} \dashrightarrow \overline{\mathbb{A}_{g,\mathbf{1}}}.$$

Let S' be the Zariski closure of $\overline{\rho}(\overline{S})$ in $\overline{\mathbb{A}_{g,\mathbf{1}}}$ and fix Weil heights $h_{\overline{S},\mathcal{M}|_{\overline{S}}}$ and $h_{S',L|_{S'}}$. Therefore, as $\dim S' = \dim \overline{S}$ and $\overline{\rho}|_{\overline{S}} : \overline{S} \dashrightarrow S'$ is dominant, Theorem 1 of [Sil11] yields positive constants ξ_7, ξ_8 and a non-empty Zariski open set $U_1 \subseteq \overline{S}$ such that

$$h_{\overline{S},\mathcal{M}|_{\overline{S}}}(s) \leq \xi_7 \cdot h_{S',L|_{S'}}(\overline{\rho}(s)) + \xi_8$$

for every $s \in U_1(\overline{\mathbb{Q}}) \subseteq \overline{S}(\overline{\mathbb{Q}})$. Since $\dim \overline{S} = 1$, U_1 is obtained by removing finitely many points from $\overline{S}$. Note also that $\overline{\rho}$ is well defined on S and it is equal to ρ. Thus, we deduce that

$$h_{\overline{S},\mathcal{M}|_{\overline{S}}}(s) \leq \xi_7 \cdot h_{\overline{\mathbb{A}_{g,\mathbf{1}}},L}(\rho(s)) + \xi_8$$

for every $s \in S(\overline{\mathbb{Q}})$. Combining this with (4.9) gives

$$h_{\overline{S},\mathcal{M}|_{\overline{S}}}(s) \leq \xi_9 \cdot h_F(s) + \xi_{10} \tag{4.10}$$

for every $s \in S(\overline{\mathbb{Q}})$ and for some positive constants ξ_9, ξ_{10}. Note that $h_F(\rho(s)) = h_F(s)$, since the Faltings height is independent of the level structure.

Now, let $\overline{\mathcal{C}}$ be the Zariski closure of $\mathcal{C}$ inside $\overline{\mathcal{A}} \subseteq \mathbb{P}^n_{\overline{\mathbb{Q}}} \times \mathbb{P}^m_{\overline{\mathbb{Q}}}$. As $\mathcal{C}$ is not contained in any fixed fiber of $\mathcal{A}$, we have that $\pi|_{\mathcal{C}} : \mathcal{C} \to S$ is surjective and thus we get a dominant rational map $\overline{\pi}|_{\overline{\mathcal{C}}} : \overline{\mathcal{C}} \dashrightarrow \overline{S}$. As above, Theorem 1 of [Sil11] yields positive constants ξ_{11}, ξ_{12} and a non-empty Zariski open set $U_2 \subseteq \overline{\mathcal{C}}$ such that

$$h_{\overline{\mathcal{C}},\overline{\mathcal{L}}|_{\overline{\mathcal{C}}}}(P) \leq \xi_{11} \cdot h_{\overline{S},\mathcal{M}|_{\overline{S}}}(\overline{\pi}(P)) + \xi_{12}$$

for every $P \in U_2(\overline{\mathbb{Q}}) \subseteq \overline{\mathcal{C}}(\overline{\mathbb{Q}})$. As before, we can assume that U_2 contains $\mathcal{C}$, so that

$$h_{\overline{\mathcal{A}},\overline{\mathcal{L}}}(P) \leq \xi_{11} \cdot h_{\overline{S},\mathcal{M}|_{\overline{S}}}(\pi(P)) + \xi_{12} \tag{4.11}$$

for every $P \in \mathcal{C}(\overline{\mathbb{Q}})$. Observe that $h_{\overline{\mathcal{C}},\overline{\mathcal{L}}|_{\overline{\mathcal{C}}}}$ is equal to the restriction of the naive height $h_{\overline{\mathcal{A}},\overline{\mathcal{L}}}$ to $\overline{\mathcal{C}}$.

Finally, by Theorem A.1 of [DGH21], there exists a positive constant ξ_{13} such that

$$\widehat{h}_{\mathcal{A}_{\pi(P)},\mathcal{L}_{\pi(P)}}(P) \leq h_{\overline{\mathcal{A}},\overline{\mathcal{L}}}(P) + \xi_{13} \cdot \max\left\{1, h_{\overline{S},\mathcal{M}|_{\overline{S}}}(\pi(P))\right\}$$

for every $P \in \mathcal{A}(\overline{\mathbb{Q}})$. Combining this with (4.10) and (4.11) we get

$$\widehat{h}_{\mathcal{A}_{\pi(P)},\mathcal{L}_{\pi(P)}}(P) \leq \xi_{14} h_F(\mathcal{A}_{\pi(P)}) + \xi_{15}$$

for some positive constants ξ_{14}, ξ_{15} and for every $P \in \mathcal{C}(\overline{\mathbb{Q}})$. □

4.7 Arithmetic bounds

Recall the setting of Theorem 4.1 and the reductions made in Section 4.3: let $S \subseteq \mathbb{A}_g = \mathbb{A}_{g,\mathbf{1},3}$ be a smooth, irreducible, locally closed curve, and let $\pi : \mathcal{A} = \mathfrak{A}_g \times_{\mathbb{A}_g} S \to S$. Let $\mathcal{C}$ be as in Theorem 4.1 and define $\mathcal{C}'$ as the set of points $P \in \mathcal{C}(\mathbb{C})$ such that $\mathcal{A}_{\pi(P)}$ has CM and there exists a nonzero endomorphism $f \in \mathrm{End}(\mathcal{A}_{\pi(P)})$ satisfying $f(P) = O_{\pi(P)}$. Equivalently, P lies in a proper algebraic subgroup of $\mathcal{A}_{\pi(P)}$.

Assume that $S, \mathcal{A}$ and $\mathcal{C}$ are defined over the same number field k. Notice that if $P \in \mathcal{C}(\mathbb{C})$, then $\mathcal{A}_{\pi(P)}$ is defined over $k(\pi(P))$ and, since π is non-constant,

$$[k(P) : k] \ll [k(\pi(P)) : k] \leq [k(P) : k] \,. \tag{4.12}$$

Moreover, since $\mathcal{C}$ is defined over $\overline{\mathbb{Q}}$ and complex abelian varieties with complex multiplication are defined over $\overline{\mathbb{Q}}$ (see Proposition 26 from Section 12.4 of [Shi98]), it follows that $\pi(P) \in S(\overline{\mathbb{Q}}) \subseteq \mathbb{A}_{g,\mathbf{1},3}(\overline{\mathbb{Q}})$ for every $P \in \mathcal{C}'$. By (4.12), this shows that $\mathcal{C}'$ is a subset of $\mathcal{C}(\overline{\mathbb{Q}})$.

From this point forward, $\gamma_1, \gamma_2, \ldots$ will be positive constants depending only on g, S, $\mathcal{A}$ and $\mathcal{C}$, unless otherwise specified.

Lemma 4.26. *Let A be a CM abelian variety of dimension g defined over a number field K. Then there exist positive constants γ_1, γ_2 depending only on g such that $h_F(A) \leq \gamma_1 \cdot [K : \mathbb{Q}]^{\gamma_2}$.*

Proof. By [Sil92], there exists a finite extension K'/K of degree at most $2 \cdot (9g)^{4g}$ such that all endomorphisms of A are defined over K'. Théorème 6.1 of [Ré17] (see also the remarks following its proof) then guarantees the existence of abelian varieties $A_1, \ldots, A_t$ defined over K' and positive integers $e_1, \ldots, e_t$ with the following properties: each A_i is $\overline{K'}$-simple, the A_i are pairwise non-isogenous over $\overline{K'}$, $\mathrm{End}_{K'}(A_i) = \mathrm{End}_{\overline{K'}}(A_i)$ is a maximal order in $\mathrm{End}^0_{\overline{K'}}(A_i)$, and A is $\overline{K'}$-isogenous to $A' := \prod_{i=1}^t A_i^{e_i}$. So, there exists an isogeny $\phi : A' \to A$ with

$$\deg \phi \leq \gamma_3 \cdot \max \left\{ h_F(A'), [K' : \mathbb{Q}] \right\}^{\gamma_4},$$

where γ_3, γ_4 are positive constants depending only on g, by [GR14a, Théorème 1.4].

Since A has CM, each A_i has CM as well, and we may consider the corresponding primitive CM types (E_i, Φ_i). Note that $\mathrm{End}_{K'}(A_i) = \mathcal{O}_{E_i}$ by construction. Then, by Corollary 3.3 of [Tsi18], there is a positive constant γ_5 depending only on g such that $h_F(A_i) \leq |\mathrm{Disc}(E_i)|^{\gamma_5}$. In addition, Theorem 4.2 of the same article yields positive constants γ_6, γ_7, again depending only on g, such that $|\mathrm{Disc}(E_i)| \leq \gamma_6 \cdot [K' : \mathbb{Q}]^{\gamma_7}$. Combining these two estimates gives

$$h_F(A_i) \leq \gamma_8 \cdot [K' : \mathbb{Q}]^{\gamma_9}$$

for some positive constants γ_8, γ_9. Since for abelian varieties A and B over a number field one has $h_F(A \times B) = h_F(A) + h_F(B)$, it follows that

$$h_F(A') = h_F\left(\prod_{i=1}^t A_i^{e_i}\right) = \sum_{i=1}^t e_i \cdot h_F(A_i) \leq \gamma_{10} \cdot [K' : \mathbb{Q}]^{\gamma_9}.$$

Applying [Fal83, Lemma 5], we deduce

$$\begin{aligned} h_F(A) &\leq h_F(A') + \frac{1}{2}\log(\deg \phi) \\ &\leq h_F(A') + \frac{\gamma_4}{2} \log \max \left\{ h_F(A'), [K' : \mathbb{Q}] \right\} + \gamma_{11} \\ &\leq \gamma_{12} \cdot [K' : \mathbb{Q}]^{\gamma_9}. \end{aligned}$$

where γ_{12} is a positive constant depending only on g.

Finally, recalling that $[K' : K] \leq 2 \cdot (9g)^{4g}$, we obtain

$$h_F(A) \leq \gamma_{12} \cdot [K' : \mathbb{Q}]^{\gamma_9} \leq \gamma_{13} \cdot [K : \mathbb{Q}]^{\gamma_{14}}$$

for suitable positive constants γ_{13}, γ_{14} depending only on g. □

Before proving the next lemma, we introduce a special $\mathbb{Z}$-basis for $\mathrm{End}(A)$. Let A be a principally polarized abelian variety of dimension g, defined over a number field K, and let † denote the Rosati involution defined by the principal polarization. By Lemma 5.1 of [MW94] and Lemma 2.1 of [MW93], there exist positive constants γ_{15}, γ_{16}, depending only on g, together with a $\mathbb{Z}$-basis $\varphi_1, \ldots, \varphi_N$ of the additive group $\mathrm{End}(A) := \mathrm{End}_{\overline{K}}(A)$ satisfying

$$\mathrm{tr}\left(\rho_r(\varphi_i^\dagger \varphi_i)\right) \leq \gamma_{15} \max\left\{[K : \mathbb{Q}], h_F(A)\right\}^{\gamma_{16}}$$

for every $i = 1, \ldots, N$. Moreover, by [BL04, Proposition 1.2.2], one has $N \leq 4g^2$.

If A is CM, then Lemma 4.26 further implies that $\varphi_1, \ldots, \varphi_N$ satisfy

$$\mathrm{tr}\left(\rho_r(\varphi_i^\dagger \varphi_i)\right) \leq \gamma_{17}\, [K : \mathbb{Q}]^{\gamma_{18}} \tag{4.13}$$

for suitable positive constants γ_{17}, γ_{18} depending only on g.

Note that, for every $s \in S(\overline{\mathbb{Q}})$, the line bundle $\mathcal{L}_s = \mathcal{L}|_{\mathcal{A}_s}$ introduced in Section 4.6, defines the same Rosati involution as the one defined by the principal polarization $\lambda_s : \mathcal{A}_s \to \widehat{A}_s$, since $\Phi_{\mathcal{L}_s} = 12\lambda_s$.

Lemma 4.27. *Let $P_0 \in \mathcal{C}'$ and define $\varphi_1, \ldots, \varphi_N \in \mathrm{End}(\mathcal{A}_{\pi(P_0)})$ as above. Then, there exists a non-zero endomorphism*

$$f_{P_0} = \sum_{i=1}^{N} a_i \varphi_i \in \mathrm{End}(\mathcal{A}_{\pi(P_0)})$$

such that $f_{P_0}(P_0) = O_{\pi(P_0)}$ and

$$\max\left\{|a_1|, \ldots, |a_N|\right\} \leq \gamma_{19}\, [k(P_0) : \mathbb{Q}]^{\gamma_{20}}$$

for some positive constants γ_{19}, γ_{20}.

Proof. Since $P_0 \in \mathcal{C}'$, there exists a non-zero $f \in \mathrm{End}(\mathcal{A}_{\pi(P_0)})$ such that $f(P_0) = O_{\pi(P_0)}$. Writing $f = \sum_{i=1}^N b_i \varphi_i$, we see that the N points $\varphi_1(P_0), \ldots, \varphi_N(P_0)$ are linearly dependent over $\mathbb{Z}$. Then, by Proposition 6.1 of [BC20] (which relies on a result by Masser [Mas88]), there exist integers $a_1, \ldots, a_N$, not all zero, together with positive constants $\gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{24}$ such that

$$f_{P_0}(P_0) = \sum_{i=1}^{N} a_i \varphi_i(P_0) = O_{\pi(P_0)}$$

and

$$\max_{1\leq i\leq N}\left\{|a_i|\right\} \leq \gamma_{21}\, [k(P_0) : \mathbb{Q}]^{\gamma_{22}} \max_{1\leq i\leq N}\left\{\widehat{h}_{\mathcal{A}_{\pi(P_0)}, \mathcal{L}_{\pi(P_0)}}(\varphi_i(P_0)), 1\right\}^{\frac{N-1}{2}} \left(h_F(\mathcal{A}_{\pi(P_0)}) + \gamma_{23}\right)^{\gamma_{24}}.$$

Here we used [MW93, Lemma 2.1] to ensure that the φ_i are defined over a finite extension of $k(\pi(P_0))$ of degree bounded by a function of g; this implies that the points $\varphi_i(P_0)$ are defined over a field of degree $\ll_g [k(P_0):\mathbb{Q}]$.

By (2.4), Corollary 2.39 and (4.13), we also have that

$$\begin{aligned}\widehat{h}_{\mathcal{A}_{\pi(P_0)},\mathcal{L}_{\pi(P_0)}}(\varphi_i(P_0)) &\leq \frac{1}{2}\operatorname{tr}\left(\rho_r(\varphi_i^\dagger\varphi_i)\right)\cdot \widehat{h}_{\mathcal{A}_{\pi(P_0)},\mathcal{L}_{\pi(P_0)}}(P_0)\\ &\leq \gamma_{25}\,[k(P_0):\mathbb{Q}]^{\gamma_{18}}\cdot \widehat{h}_{\mathcal{A}_{\pi(P_0)},\mathcal{L}_{\pi(P_0)}}(P_0).\end{aligned}$$

Moreover, Proposition 4.25 shows that $\widehat{h}_{\mathcal{A}_{\pi(P_0)},\mathcal{L}_{\pi(P_0)}}(P_0)\leq \gamma_{26}h_F(\mathcal{A}_{\pi(P_0)})+\gamma_{27}$. Hence,

$$\widehat{h}_{\mathcal{A}_{\pi(P_0)},\mathcal{L}_{\pi(P_0)}}(\varphi_i(P_0))\leq \gamma_{25}\,[k(P_0):\mathbb{Q}]^{\gamma_{18}}\cdot(\gamma_{26}h_F(\mathcal{A}_{\pi(P_0)})+\gamma_{27}).$$

Since $N\leq 4g^2$ and $\mathcal{A}_{\pi(P_0)}$ has CM, this implies

$$\begin{aligned}\max_{1\leq i\leq N}\{|a_i|\} &\leq \gamma_{21}\,[k(P_0):\mathbb{Q}]^{\gamma_{22}}\cdot\max_i\left\{\widehat{h}_{\mathcal{A}_{\pi(P_0)},\mathcal{L}_{\pi(P_0)}}(\varphi_i(P_0)),1\right\}^{\frac{N-1}{2}}\left(h_F(\mathcal{A}_{\pi(P_0)})+\gamma_{23}\right)^{\gamma_{24}}\\ &\leq \gamma_{28}\,[k(P_0):\mathbb{Q}]^{\gamma_{29}}\cdot\left(\gamma_{26}h_F(\mathcal{A}_{\pi(P_0)})+\gamma_{27}\right)^{2g^2}\left(h_F(\mathcal{A}_{\pi(P_0)})+\gamma_{23}\right)^{\gamma_{24}}\\ &\leq \gamma_{30}\,[k(P_0):\mathbb{Q}]^{\gamma_{31}}\end{aligned}$$

by Lemma 4.26. □

Now, let $P_0\in\mathcal{C}'$ and choose $\tau_{P_0}\in u_b^{-1}(\pi(P_0))\cap\mathfrak{F}_\Gamma$, where $\Gamma=\Gamma_{\mathbf{1},3}$, $\mathfrak{F}_\Gamma$ and the uniformization map $u_b:\mathbb{H}_g\to\mathbb{A}_{g,\mathbf{1},3}(\mathbb{C})$ were introduced in Section 4.2.1. The set $u_b^{-1}(\pi(P_0))\cap\mathfrak{F}_\Gamma$ contains a single element unless some preimage of $\pi(P_0)$ lies on the boundary of $\mathfrak{F}_\Gamma$, in which case it contains $O(g)$ elements.

Let $Z_{P_0}\in\mathfrak{F}_g$ be a point in the $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit of τ_{P_0}. Then one can choose a symplectic basis of the period lattice of $\mathcal{A}_{\pi(P_0)}$ such that the corresponding period matrix is $(Z_{P_0},\mathbf{1})$, once the level structure is disregarded.[1] In the sequel, we fix this symplectic basis, and all analytic and rational representations of endomorphisms of $\mathcal{A}_{\pi(P_0)}$ will be defined with respect to it.

Since $\mathcal{A}_{\pi(P_0)}$ has CM, it is known (see for instance Section 6.2 of [Tsi18] or [Shi92]) that

$$[\mathbb{Q}(Z_{P_0}):\mathbb{Q}]\leq 2g.$$

Moreover, if we write $\tau_{P_0}=\sigma\cdot Z_{P_0}$ for some $\sigma\in\mathrm{Sp}_{2g}(\mathbb{Z})$, then we easily see that $\mathbb{Q}(\tau_{P_0})\subseteq\mathbb{Q}(Z_{P_0})$, since σ has integer entries.

We now establish bounds for the heights of τ_{P_0} and Z_{P_0}.

Lemma 4.28. *Let $P_0\in\mathcal{C}'$ and let τ_{P_0} and Z_{P_0} be as above. Then, there are positive constants γ_{32}, γ_{33}, γ_{34}, γ_{35}, such that $H_{\max}(Z_{P_0})\leq\gamma_{32}\cdot[k(P_0):\mathbb{Q}]^{\gamma_{33}}$ and $H_{\max}(\tau_{P_0})\leq\gamma_{34}\cdot[k(P_0):\mathbb{Q}]^{\gamma_{35}}$,*

[1] If the level structure is taken into account, then one can choose a symplectic basis so that the period matrix is $(\tau_{P_0},\mathbf{1})$.

where $H_{\max}$ is the entry-wise height on $\mathrm{Mat}_g(\overline{\mathbb{Q}})$ defined in Section 2.3.1.

Proof. Since $\mathcal{A}_{\pi(P_0)}$ has CM, Z_{P_0} is a CM point in $\mathfrak{F}_g$. Thus, by Theorem 1.3 of [PT13] together with Theorem 5.2 of [Tsi18], there exist positive constants $\gamma_{36}, \gamma_{37}, \gamma_{38}, \gamma_{39}$, depending only on g, such that

$$H_{\max}(Z_{P_0}) \leq \gamma_{36} \cdot \# \left(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot \pi(P_0)\right)^{\gamma_{37}} \leq \gamma_{38} \cdot [k(P_0) : \mathbb{Q}]^{\gamma_{39}}. \tag{4.14}$$

Now, take $\sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $\tau_{P_0} = \sigma \cdot Z_{P_0} = (AZ_{P_0} + B)(CZ_{P_0} + D)^{-1}$. Recall that the definition of $\mathfrak{F}_\Gamma$ (see (4.1)) implies that we can take σ to be one of the chosen representatives $\sigma_1, \ldots, \sigma_n$ for the right cosets of Γ in $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Then, using Proposition 2.18, we get

$$\begin{aligned} H_{\max}(\tau_{P_0}) &\leq g \cdot H_{\max}(AZ_{P_0} + B)^g \cdot H_{\max}\left((CZ_{P_0} + D)^{-1}\right)^g \\ &\ll_g H_{\max}(AZ_{P_0})^g H_{\max}(B)^g \cdot H_{\max}(CZ_{P_0})^{2g^4 - g^3} H_{\max}(D)^{2g^4 - g^3} \\ &\ll_g H_{\max}(A)^{g^2} H_{\max}(B)^g H_{\max}(C)^{2g^5 - g^4} H_{\max}(D)^{2g^4 - g^3} \cdot H_{\max}(Z_{P_0})^{2g^5 - g^4 + g^2} \end{aligned}$$

This implies that there exist a constant γ_{40}, depending only on g and σ, such that

$$H_{\max}(\tau_{P_0}) \leq \gamma_{40} H_{\max}(Z_{P_0})^{2g^5 - g^4 + g^2}.$$

Taking the maximum of all such constants over all possible choices of $\sigma \in \{\sigma_1, \ldots, \sigma_n\}$, we get a constant γ_{41} that depends only on g and the choice of $\sigma_1, \ldots, \sigma_n$, such that

$$H_{\max}(\tau_{P_0}) \leq \gamma_{41} H_{\max}(Z_{P_0})^{2g^5 - g^4 + g^2}.$$

Finally, substituting the bound (4.14) for $H_{\max}(Z_{P_0})$, gives the desired bound for $H_{\max}(\tau_{P_0})$. □

Lemma 4.29. *Let $P_0 \in \mathcal{C}'$ and f_{P_0} be the endomorphism given by Lemma 4.27. Then, $\rho_a(f_{P_0}) \in \mathrm{Mat}_g(\mathbb{C})$ has algebraic entries and $H_{2g}\left(\rho_a(f_{P_0})\right) \leq \gamma_{42} \cdot [k(P_0) : \mathbb{Q}]^{\gamma_{43}}$, for some positive constants γ_{42}, γ_{43}.*

Proof. Write $\rho_r(f_{P_0}) = \left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & M_4 \end{smallmatrix}\right)$, where $M_\ell = \left(m_{i,j}^{(\ell)}\right)_{1 \leq i,j \leq g} \in \mathrm{Mat}_g(\mathbb{Z})$ for $\ell = 1, 2, 3, 4$. Then, by Equation (2.1), $\rho_a(f_{P_0}) = Z_{P_0} M_2 + M_4$, as $\mathcal{A}_{\pi(P_0)}$ is principally polarized by assumption. This proves that $\rho_a(f_{P_0}) \in \mathrm{Mat}_g(\mathbb{Q}(Z_{P_0})) \subseteq \mathrm{Mat}_g(\overline{\mathbb{Q}})$. Note also that all entries of $\rho_a(f_{P_0})$ have degree at most $2g$.

Hence, Proposition 2.18 implies

$$\begin{aligned} H_{\max}(\rho_a(f_{P_0})) &\leq 2H_{\max}(Z_{P_0} M_2) H_{\max}(M_4) \leq 2g H_{\max}(Z_{P_0})^g H_{\max}(M_2)^g H_{\max}(M_4) \\ &\leq 2g \left\| \rho_r(f_{P_0}) \right\|_\infty^{g+1} H_{\max}(Z_{P_0})^g \end{aligned}$$

and, by Lemma 4.7,

$$H_{2g}(\rho_a(f_{P_0})) \leq 2^{2g} \cdot H_{\max}(\rho_a(f_{P_0}))^{2g} \leq (4g)^{2g} \cdot \left\|\rho_r(f_{P_0})\right\|_\infty^{2g(g+1)} \cdot H_{\max}(Z_{P_0})^{2g^2}.$$

Furthermore, since $f_{P_0} = \sum_{i=1}^N a_i\varphi_i$, we also have

$$\begin{aligned}\left\|\rho_r(f_{P_0})\right\|_\infty &= \left\|\sum_{i=1}^N a_i\rho_r(\varphi_i)\right\|_\infty \leq \sum_{i=1}^N |a_i| \cdot \|\rho_r(\varphi_i)\|_\infty \\ &\leq N \cdot \max\{|a_1|, \dots, |a_N|\} \cdot \max_i \{\|\rho_r(\varphi_i)\|_\infty\}.\end{aligned}$$

By Proposition 4.15, there are positive constants γ_{44}, γ_{45} such that

$$\|\rho_r(\varphi_i)\|_\infty \leq \gamma_{44} \cdot \max\{1, \|\mathrm{Im}(Z_{P_0})\|_\infty\}^{\gamma_{45}} \cdot \sqrt{\mathrm{tr}\left(\rho_r(\varphi_i^\dagger\varphi_i)\right)}.$$

We then use Lemma 4.7 and Lemma 4.28 to get

$$\begin{aligned}\|\mathrm{Im}(Z_{P_0})\|_\infty &\leq \|Z_{P_0}\|_\infty \leq \sqrt{2g+1} \cdot H_{2g}(Z_{P_0}) \\ &\leq 2^{2g}\sqrt{2g+1} \cdot H_{\max}(Z_{P_0})^{2g} \leq \gamma_{46} \cdot [k(P_0) : \mathbb{Q}]^{\gamma_{47}}\end{aligned} \tag{4.15}$$

which, combined with (4.13), implies that

$$\|\rho_r(\varphi_i)\|_\infty \leq \gamma_{48} \cdot [k(P_0) : \mathbb{Q}]^{\gamma_{49}}.$$

Moreover, we use Lemma 4.27 to bound $\max\{|a_1|, \dots, |a_N|\}$, so that

$$\left\|\rho_r(f_{P_0})\right\|_\infty \leq 4g^2 \cdot \max\{|a_1|, \dots, |a_N|\} \cdot \max_i\{\|\rho_r(\varphi_i)\|_\infty\} \leq \gamma_{50} \cdot [k(P_0) : \mathbb{Q}]^{\gamma_{51}}.$$

Finally, we get

$$\begin{aligned}H_{2g}(\rho_a(f_{P_0})) &\leq (4g)^{2g} \cdot \left\|\rho_r(f_{P_0})\right\|_\infty^{2g(g+1)} \cdot H_{\max}(Z_{P_0})^{2g^2} \\ &\leq (4g)^{2g} \cdot \gamma_{50} \cdot [k(P_0) : \mathbb{Q}]^{\gamma_{51}} \cdot \gamma_{32}^{2g^2} \cdot [k(P_0) : \mathbb{Q}]^{2g^2\gamma_{33}} \\ &\leq \gamma_{52} \cdot [k(P_0) : \mathbb{Q}]^{\gamma_{53}}\end{aligned}$$

by Lemma 4.28. □

Lemma 4.30. *Let $P_0 \in \mathcal{C}'$ and let τ_{P_0} be as above. Then, there are positive constants γ_{54}, γ_{55} such that*

$$\det(\mathrm{Im}(\tau_{P_0})) \geq \frac{\gamma_{54}}{[k(P_0) : \mathbb{Q}]^{\gamma_{55}}}.$$

Proof. By Proposition 4.20, we have that $\det(\mathrm{Im}(\tau_{P_0})) \geq \delta_3 \max\{1, \|\mathrm{Im}(Z_{P_0})\|_\infty\}^{-2g}$. Hence, (4.15) implies that

$$\det(\operatorname{Im}(\tau_{P_0})) \geq \frac{\delta_3}{\max\{1, \|\operatorname{Im}(Z_{P_0})\|_\infty\}^{2g}} \geq \frac{\delta_3}{\gamma_{46}^{2g} \cdot [k(P_0) : \mathbb{Q}]^{2g\gamma_{47}}}$$

which gives the desired bound. □

4.8 Proof of Theorem 4.1

We need to establish the finiteness of the set $\mathcal{C}'$, introduced at the beginning of the previous section.

Let $P_0 \in \mathcal{C}'$ and let $\sigma \in \operatorname{Gal}(\overline{k}/k)$. We aim to show that $\sigma(P_0) \in \mathcal{C}'$.

Since the abelian varieties $\mathcal{A}_{\pi(\sigma(P_0))}$ and $\mathcal{A}_{\pi(P_0)}$ have isomorphic endomorphism rings, it follows that both are CM abelian varieties. Moreover, the action of σ sends subgroups of $\mathcal{A}_{\pi(P_0)}$ to subgroups of $\mathcal{A}_{\pi(\sigma(P_0))}$, preserving their dimensions. Consequently, if P_0 is contained in a proper algebraic subgroup of $\mathcal{A}_{\pi(P_0)}$, then $\sigma(P_0)$ must be also contained in a proper algebraic subgroup of $\mathcal{A}_{\pi(\sigma(P_0))}$. Thus, $\sigma(P_0) \in \mathcal{C}'$.

To simplify notation, we set $d_0 := [k(P_0) : \mathbb{Q}] = [k(\sigma(P_0)) : \mathbb{Q}]$. Then, Lemma 4.27 and Lemma 4.29 imply the existence of a nonzero endomorphism $f_{\sigma(P_0)} \in \operatorname{End}\left(\mathcal{A}_{\pi(\sigma(P_0))}\right)$ such that

$$f_{\sigma(P_0)}\left(\sigma(P_0)\right) = O_{\pi(\sigma(P_0))} \quad \text{and} \quad H_{2g}\left(\rho_a(f_{\sigma(P_0)})\right) \leq \gamma_{42} \cdot d_0^{\gamma_{43}}.$$

Moreover, combining Lemmas 4.7 and 4.28 yields

$$H_{2g}(\tau_{\sigma(P_0)}) \leq 2^{2g} \cdot H_{\max}(Z_{P_0})^{2g} \leq \gamma_{56} \cdot d_0^{\gamma_{57}}.$$

In addition, Lemma 4.30 gives the lower bound

$$\det(\operatorname{Im}(\tau_{\sigma(P_0)})) \geq \frac{\gamma_{54}}{d_0^{\gamma_{55}}}.$$

Hence, as σ varies in $\operatorname{Gal}(\overline{k}/k)$, the elements of $u^{-1}(\sigma(P_0)) \cap \mathcal{F}_g$ are all contained in the set $\mathcal{Z}(\gamma d_0^{\eta})$, where $\mathcal{Z}(T)$ is the set defined at the start of Section 4.5, with $\gamma = \max\left\{\gamma_{42}, \gamma_{56}, \frac{1}{\gamma_{54}}\right\}$ and $\eta = \max\{\gamma_{43}, \gamma_{57}, \gamma_{55}\}$.

However, the argument above implies that there are at least $d_0/[k : \mathbb{Q}]$ distinct points in $u^{-1}(\sigma(P_0)) \cap \mathcal{F}_g$ that are contained in $\mathcal{Z}(\gamma d_0^{\eta})$. Applying Proposition 4.22 with $\varepsilon = \frac{1}{2\eta}$, we deduce that d_0 is uniformly bounded for all $P_0 \in \mathcal{C}'$.

Hence, by Lemma 4.26, the Faltings height $h_F(\mathcal{A}_{\pi(P_0)})$ is bounded above by a constant independent of $P_0 \in \mathcal{C}'$. In view of (4.10), it follows that the height $h_{\overline{S},\mathcal{M}|_{\overline{S}}}$ is bounded on $\pi(\mathcal{C}') \subseteq S(\overline{\mathbb{Q}})$. Consequently, $\pi(\mathcal{C}') \subseteq \overline{S}(\overline{\mathbb{Q}})$ is a set of bounded height and bounded degree, as $[k(\pi(P_0)) : \mathbb{Q}] \leq d_0$. Since $\mathcal{M}|_{\overline{S}}$ is ample, the Northcott property of the Weil height (part (6) of Theorem 2.22) ensures that $\pi(\mathcal{C}')$ is finite.

Therefore, $\mathcal{C}'$ is contained in the intersection of $\mathcal{C}$ with the union of finitely many fibers of $\mathcal{A} \to S$. As $\mathcal{C}$ is irreducible and not contained in any fiber, we conclude that $\mathcal{C}'$ itself is finite.

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